NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM

No. 1196

NONSTATIONARY GAS FLOW IN THIN PIPES OF VARIABLE CROSS SECTION

By G. Guderley

Translation of ZWB, Forschungsbericht No. 1744, October 1942

Washington
December 1948

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ABSTRACT

Characteristic methods for nonstationary flows have been published only for the special case of the isentropic flow up until the present, although they are applicable in various places to more difficult questions, too. The present report derives the characteristic method for the flows which depend only on the position coordinates and the time. At the same time the treatment of compression shocks is shown. To simplify the application numerous examples are worked out.

*"Nichtstationäre Gasströmungen in dünnen Rohren veränderlichen Querschnitts." Zentrale für wissenschaftliches Berichtswesen der Luftfahrtforschung des Generalluftzeugmeisters (ZWB) Berlin-Adlershof, Forschungsbericht Nr. 1744, Braunschweig, Oct. 22, 1942

1. INTRODUCTION

In papers by F. Schultz-Grunow and R. Sauer[1] methods have been developed recently for completely solving the problem of nonstationary isentropic gas flows in a pipe of constant cross section. An expanded view of the problem is the basis for the present report. Flows are considered, which likewise depend only on the position coordinate; however, the cross section of the tube need no longer be constant and the entropy may vary from particle to particle. The method of solution applied here has been discovered almost simultaneously in several places, by Adam Schmidt, W. Döring, and F. Pfeiffer, among others.

The application of the characteristic method is possible without a previous substantial knowledge of mathematics. Correspondingly, if a derivation was desired too, one could be had which did not make any special mathematical demands on the reader. As a model, the Busemann derivation of the characteristic method for two-dimensional stationary gas flows might possibly do[2]. It is actually possible to apply this derivation immediately to the

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[1]Schultz-Grunow, F.: Nichtstationäre eindimensionale Gasbewegung. Forschung auf dem Gebiet des Ingenieurwesens, Bd. 13 (1942) pp. 125 to 134. Sauer, R.: Charakteristikenverfahren für die eindimensionale instationäre Gasströmung, Ingenieur-Archiv, XIII Vol. (1942) pp. 79 to 89. Vorbereitende Untersuchungen sowie Anwendungen finden sich in den Arbeiten von H. Pfriem. Zur Theorie ebener Druckwellen mit steiler Front Akustische Zeitschrift Jahrg. 6 (1941) part 4. – Die ebene ungedämpfte Druckwelle grosser Schwingungsweite, Forschung Vol. 12 (194) pp. 51 to 64 – Reflexionsgesetze für ebene Druckwellen grosser Schwingungsweite, Forschung Vol. 12 (1941) pp. 244 to 256 – Zur gegenseitigen Überlagerung ungedämpfter ebener Gasswellen grosser Schwingungsweite, Akustische Zeitschrift Jahrg. 7 (1942) part 2 – Zur Frage der oberen Grenze von Geschossgeschwindigkeiten Zeitschrift f. techn. Physik 22 (1941) pp. 255 to 260. Eine weitere Anwendung findet sich bei G. Damköhler und A. Schmidt, Gasdynamische Beiträge zur Auswertung von Flammenversuchen in Rohrstrecken. Zeitschrift für Elektrotechnik Vol. 47 (1941) pp. 547 to 567.

[2]Busemann, A.: Beitrag Gasdynamik in Handbuch der Experimentalphysik (Wien-Harms) Bd. 4, Teil 1, p. 421 and adjoining pages.

isentropic nonstationary flow in a pipe of constant cross section and from this by means of some supplementary physical concepts succeed in getting a treatment of flows in a tube of variable cross section; this is the course which had been taken, originally. In comparison to the mathematical theory of characteristics, however, these considerations operate with a lack of clarity sufficient so that the mathematical theory - for the engineer too - can be represented as the best approach to the characteristics method.

Considerations necessary for the present problem are now brought forward from the characteristics theory[3]. As a result, equations for the directions of the characteristics as well as conditions which must be satisfied along the characteristics are obtained. Proceeding from these relations, the next sections develop the actual method of computation. Next, the characteristics method for the case which is familiar by now, that of isentropic flows in a pipe of constant cross section, is deduced again and the transformations appearing there are used to simplify the computation in complicated cases, too. Since this is not always possible, the most general form of the characteristics method is shown in a later section. After this, the formulas obtained for the special case of an ideal gas with constant specific heat are simplified and the consideration of boundary conditions explained. The remaining sections deal with calculation of compression shocks; the known relations which connect the phase before and behind a compression shock with one another are set forth in a convenient form for the present problems and with that the calculation of a compression shock in a flow is carried out.

The theory is illustrated with suitable examples treated in detail. In that regard, it seemed advantageous to avoid definite problems of technical interest, in doing so gaining the possibility of working out examples under very general assumptions without excessive effort. It is hoped that, nevertheless, the application of the method to physical problems offers no additional difficulties worth mentioning inasmuch as the earlier publications contain such applications. The author expresses his thanks to Dr. Hans Lehmann for working out the examples.

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[3]Compare Courant-Hilbert, "Methoden der mathematischen Physik II", p 291. Guderley follows the representation given by H. Seifert at the same institute for Gas Dynamics in lectures.

2. BASIC EQUATIONS

Consider nonstationary, perfect gas flows in a pipe with a cross-section that varies in space and time[4] in the neighborhood of the flow tube; that is, it is assumed that the velocity and the phase over a cross section of the pipe may be considered as sufficiently constant. In general, this assumption is justifiable only if the thickness of the tube, relative to its length, changes slowly enough. Only for flows which have as surfaces of constant phase parallel planes, coaxial cylinders or concentric spheres need this limitation be ignored. These flows with plane, cylindrical, or spherical wave propagation are included as special cases in the present problem.

To stress the relationship to stationary two-dimensional flows, let the axis of the pipe be vertical, the position coordinate be y, the time be t and plotted horizontally. In this yt-diagram the flows are investigated. (Compare fig. 1.)

Let

p pressure

s entropy per unit mass

ρ density

v velocity

$F = F(yt)$ the cross section of the pipe, let F be given

In a region free of compression shocks, the flow is described by the dependency of the density on pressure and entropy, the Newtonian principle, the equation of continuity and the statement that the entropy of a particle is preserved, as follows:

$$\rho = \rho(s,p) \tag{1}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} + v\frac{\partial v}{\partial y} + \frac{\partial v}{\partial t} = 0 \tag{2}$$

$$v\frac{\partial \rho}{\partial y} + \rho\frac{\partial v}{\partial y} + \frac{\partial \rho}{\partial t} + \rho v\frac{\partial \ln F}{\partial y} + \rho\frac{\partial \ln F}{\partial t} = 0 \tag{3}$$

$$v\frac{\partial s}{\partial y} + \frac{\partial s}{\partial t} = 0 \tag{4}$$

[4]Problems with time variations in the cross-sectional area are rare; they were included, since they can be handled without additional difficulty.

In this the derivatives of ρ should be replaced by the derivatives of p and s, for this purpose

$$\frac{\partial \rho}{\partial p} = \frac{1}{a^2} \tag{5}$$

is introduced. Therefore, instead of equation (3)

$$\frac{v}{a^2}\frac{\partial p}{\partial y} + v\frac{\partial \rho}{\partial s}\frac{\partial s}{\partial y} + \rho\frac{\partial v}{\partial y} + \frac{1}{a^2}\frac{\partial p}{\partial t} + \frac{\partial \rho}{\partial s}\frac{\partial s}{\partial t} + \rho v\frac{\partial \ln F}{\partial y} + \rho\frac{\partial \ln F}{\partial t} = 0 \tag{3a}$$

is obtained.

3. FROM THE THEORY OF CHARACTERISTICS

In regard to the system of equations (2), (3a), and (4), the familiar question is raised from the theory of characteristics. In a region of the yt-plane let the solution of this system of equations and its derivations be finite throughout. On a curve C placed in this range let the values p, v, and s which correspond to this solution and, therefore, the appropriate derivatives taken in the direction of C be known. The question is asked whether the derivatives in other directions may be computed with the aid of the system of differential equations, and under what conditions. To answer this, a curvilinear coordinate system ξ,η is introduced in which a curve ξ = constant coincides with C (fig. 1). All the derivatives with respect to η along this curve ξ = constant are given, the derivatives with respect to ξ are sought. This transformation is carried out and terms are arranged so that the unknown derivatives with respect to ξ are on the left and only known quantities are on the right. That is

$$\xi = \xi(y,t)$$

$$\eta = \eta(y,t)$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t}$$

With this form (2), (3(a)), and (4) are obtained

$$
\left.\begin{aligned}
&\frac{\partial p}{\partial \xi}\frac{1}{\rho}\frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \xi}\left(v\frac{\partial \xi}{\partial y} + \frac{\partial \xi}{\partial t}\right) = -\frac{\partial p}{\partial \eta}\frac{1}{\rho}\frac{\partial \eta}{\partial y} - \frac{\partial v}{\partial \eta}\left(v\frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial t}\right) \\
&\frac{\partial p}{\partial \xi}\frac{1}{a^2}\left(v\frac{\partial \xi}{\partial y} + \frac{\partial \xi}{\partial t}\right) + \frac{\partial v}{\partial \xi}\rho\frac{\partial \xi}{\partial y} + \frac{\partial s}{\partial \xi}\frac{\partial \rho}{\partial s}\left(v\frac{\partial \xi}{\partial y} + \frac{\partial \xi}{\partial t}\right) \\
&\quad = -\frac{\partial p}{\partial \eta}\frac{1}{a^2}\left(v\frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial t}\right) - \frac{\partial v}{\partial \eta}\rho\frac{\partial \eta}{\partial y} - \frac{\partial s}{\partial \eta}\left(v\frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial t}\right)\frac{\partial \rho}{\partial s} - \rho v\frac{\partial \ln F}{\partial y} - \rho\frac{\partial \ln F}{\partial t} \\
&\frac{\partial s}{\partial \xi}\left(v\frac{\partial \xi}{\partial y} + \frac{\partial \xi}{\partial t}\right) = -\frac{\partial s}{\partial \eta}\left(v\frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial t}\right)
\end{aligned}\right\} \quad (6)
$$

A linear system of equations is obtained for the unknowns $\frac{\partial p}{\partial \xi}$, $\frac{\partial v}{\partial \xi}$, and $\frac{\partial s}{\partial \xi}$; the unknowns, themselves, are obtained by Cramer's rule as the quotient of two determinants. The determinant in the denominator is the same for all unknowns. It always gives a single-valued solution for the system of equations, if the determinant in the denominator is different from 0. In the other case with a determinant in the denominator that vanishes, it is a necessary condition for the existence of solutions that remain finite that the determinants in the numerator also vanish. In this case, however, the solution of the system of equations is only defined over any portion of the solution of the homogenous system. In application to the system of equations (6) signifies the following: The determinant in the denominator is formed from the coefficients of the unknowns. Considering a fixed point on C, at which p, v, and s are known by assumption, the coefficients depend on $\frac{\partial \xi}{\partial y}$ and $\frac{\partial \xi}{\partial t}$, that is the direction of C. If C is so directed that the determinant in the denominator does not vanish anywhere, the $\frac{\partial p}{\partial \xi}$ etc. are computable as single valued.

Of greater interest for our considerations is the other case, namely, that the determinant in the denominator is zero at every point of C. Such a curve is termed a characteristic. Because of the assumption of finite derivatives the determinants in the numerator also vanish. Relations are obtained thereby, in which the right side of (6) and, therefore, the derivatives of p, v, and s along C appear as essential ingredients. These relations represent the starting point of the graphical numerical method of solution.

Since the solutions of a linear system of equations are no longer single valued for vanishing numerator and denominator determinants, the pursuit of a given solution of a characteristic is possible in various ways. These different possibilities actually appear on changing the initial and boundary conditions.

4. THE DIRECTIONS OF THE CHARACTERISTICS

To find the directions for which the curve ξ = constant is a characteristic, the determinant in the denominator must be set equal to zero in the solutions of the system of equations (6).

$$\begin{vmatrix} \frac{1}{\rho}\frac{\partial\xi}{\partial y} & v\left(\frac{\partial\xi}{\partial y}+\frac{\partial\xi}{\partial t}\right) & 0 \\ \frac{1}{a^2}\left(v\frac{\partial\xi}{\partial y}+\frac{\partial\xi}{\partial t}\right) & \rho\frac{\partial\xi}{\partial y} & \frac{\partial\rho}{\partial s}\left(v\frac{\partial\xi}{\partial y}+\frac{\partial\xi}{\partial t}\right) \\ 0 & 0 & \left(v\frac{\partial\xi}{\partial y}+\frac{\partial\xi}{\partial t}\right) \end{vmatrix} = 0$$

This gives

$$\left(v\frac{\partial\xi}{\partial y}+\frac{\partial\xi}{\partial t}\right)\left[\left(\frac{\partial\xi}{\partial y}\right)^2-\frac{1}{a^2}\left(v\frac{\partial\xi}{\partial y}+\frac{\partial\xi}{\partial t}\right)^2\right]=0$$

From this are obtained the conditions

$$v\frac{\partial\xi}{\partial y}+\frac{\partial\xi}{\partial t}=0 \tag{7}$$

or

$$\left(v+a\right)\frac{\partial\xi}{\partial y}+\frac{\partial\xi}{\partial t}=0 \tag{8a}$$

or

$$(v - a)\frac{\partial\xi}{\partial y} + \frac{\partial\xi}{\partial t} = 0 \tag{8b}$$

The slope of any curve ξ = constant is given by

$$\frac{dy}{dt} = -\frac{\frac{\partial\xi}{\partial t}}{\frac{\partial\xi}{\partial y}}$$

From (7) and (8), together with this, the slopes of the characteristics are

$$\frac{dy}{dt} = v \tag{9}$$

or

$$\frac{dy}{dt} = v + a \tag{10a}$$

or

$$\frac{dy}{dt} = v - a \tag{10b}$$

The characteristics defined by (9) are path-time curves for the individual gas particle; they might be termed life lines of the particles. According to (10), velocities are determined from the slope of the other characteristics, which differ from the velocity of the particles by $\pm a$. For stationary flows the Mach waves correspond to these last characteristics; this designation will be adopted. Therefore, let Mach waves of the first family be those which spread out with the velocity $v + a$ and Mach waves of the second family be associated with the velocity $v - a$.

5. THE CONSISTENCY CONDITIONS ON THE CHARACTERISTICS

As shown in section 3, along the characteristics, certain conditions must be complied with by the derivatives which result from the vanishing of the determinant in the numerator. These conditions are called consistency conditions, for p, v, and s are subject to them, if the derivatives with respect to ξ are to remain finite. If the right side of (6) is designated R_1, R_2, and R_3 in sequence, then the following is obtained for the determinant in the numerator of the quotient for $\frac{\partial p}{\partial \xi}$:

$$\begin{vmatrix} R_1 & v\frac{\partial \xi}{\partial y} + \frac{\partial \xi}{\partial t} & 0 \\ R_2 & \rho\frac{\partial \xi}{\partial y} & \frac{\partial \rho}{\partial s}\left(v\frac{\partial \xi}{\partial y} + \frac{\partial \xi}{\partial t}\right) \\ R_3 & 0 & \left(v\frac{\partial \xi}{\partial y} + \frac{\partial \xi}{\partial t}\right) \end{vmatrix} = 0 \qquad (11)$$

This determinant must vanish to give the directions of the characteristics. Substituting (7) gives

$$\begin{vmatrix} R_1 & 0 & 0 \\ R_2 & \rho\frac{\partial \xi}{\partial y} & 0 \\ R_3 & 0 & 0 \end{vmatrix} = 0$$

According to this the determinant (11) vanishes by itself. With (8a), that is, for a Mach wave 1

$$\begin{vmatrix} R_1 & -a\frac{\partial \xi}{\partial y} & 0 \\ R_2 & \rho\frac{\partial \xi}{\partial y} & -\frac{\partial \rho}{\partial s}a\frac{\partial \xi}{\partial y} \\ R_3 & 0 & -a\frac{\partial \xi}{\partial y} \end{vmatrix} = 0$$

is obtained, or

$$\left(\frac{\partial \xi}{\partial y}\right)^2 \left[-\rho a R_1 - a^2 R_2 + \frac{\partial \rho}{\partial s} a^2 R_3\right] = 0$$

In this, $\frac{\partial \xi}{\partial y}$ is certainly different from zero, as long as v and a are finite and grad $\xi \neq 0$. As the condition for the Mach wave 1 is obtained

$$-\rho R_1 - a R_2 + a \frac{\partial \rho}{\partial s} R_3 = 0 \tag{12a}$$

The consistency conditions for the Mach waves 2 is, if a is replaced by $-a$

$$-\rho R_1 + a R_2 - a \frac{\partial \rho}{\partial s} R_3 = 0 \tag{12b}$$

A condition for the life line is obtained if the vanishing of the determinant in the numerator in the quotient for $\frac{\partial s}{\partial \xi}$ to be got from (6) requires

$$\begin{vmatrix} \frac{1}{\rho} \frac{\partial \xi}{\partial y} & v\left(\frac{\partial \xi}{\partial y} + \frac{\partial \xi}{\partial t}\right) & R_1 \\ \frac{1}{a^2}\left(v \frac{\partial \xi}{\partial y} + \frac{\partial \xi}{\partial t}\right) & \rho \frac{\partial \xi}{\partial y} & R_2 \\ 0 & 0 & R_3 \end{vmatrix} = 0$$

For the Mach wave this equation is satisfied by itself, the condition for the life line is

$$R_3 = 0 \tag{13}$$

The determinant in the numerator of $\frac{\partial v}{\partial \xi}$ could be investigated, too; however, this would not give any new consistency conditions.

The values of R_1, R_2, and R_3 are still to be put in. From (12a),

$$\frac{1}{a}\frac{\partial p}{\partial \eta}\left[\frac{\partial \eta}{\partial y}(v + a) + \frac{\partial \eta}{\partial t}\right] + \rho \frac{\partial v}{\partial \eta}\left[\frac{\partial \eta}{\partial y}(v + a) + \frac{\partial \eta}{\partial t}\right] = -a\rho v \frac{\partial \ln F}{\partial y} - a \frac{\partial \ln F}{\partial t}$$

is obtained. The direction $\frac{dy}{dt}$ along a Mach wave 1 is given by (10a); on that account

$$\frac{d\eta}{dt} = \frac{\partial \eta}{\partial y}\frac{dy}{dt} + \frac{\partial \eta}{\partial t} = \frac{\partial \eta}{\partial y}(v + a) + \frac{\partial \eta}{\partial t}$$

is valid for it. With that the consistency condition for the Mach wave 1 can be written in the form

$$\frac{1}{a\rho}\frac{dp}{dt} + \frac{dv}{dt} = -a\left(v\frac{\partial \ln F}{\partial y} + \frac{\partial \ln F}{\partial t}\right) \qquad (14a)$$

The consistency condition for a Mach wave 2 is obtained, by substituting $-a$ for a

$$\frac{1}{a\rho}\frac{dp}{dt} - \frac{dv}{dt} = -a\left(v\frac{\partial \ln F}{\partial y} + \frac{\partial \ln F}{\partial t}\right) \qquad (14b)$$

From (13) for a life line is obtained

$$\frac{\partial s}{\partial \eta} = 0$$

This may be integrated immediately

$$s = \text{constant} \qquad (15)$$

Naturally this constant will differ from particle to particle, in general.

6. FLOWS WITH CONSTANT ENTROPY AND UNIFORM PIPE CROSS SECTION

Equations (14), (15), (9), and (10) just obtained are certainly useful, fundamentally, as a starting point of a characteristics method - in fact, there are examples, where it is necessary to revert to them (compare section 11); in most cases, however, there are still other transformations suitable. The direction in which to proceed for these is obtained if an attempt is made to derive the characteristic method for isentropic flows in a pipe of uniform cross section from equations (14) and (15) possibly in the form applied by Schultz-Grunow. To emphasize the fundamental ideas, no assumptions of any kind are made therein of the characteristics of the flow medium.

On account of the hypothesis of constant entropy, equation (15) satisfies itself. In equations (14) the right sides are omitted since it concerns a tube of uniform cross section. Further, on account of the hypothesis of constant entropy the state of the gas is still dependent as only one variable, perhaps the pressure, or the temperature; the quantities appearing on the left side and a are accordingly functions of this variable. It is possible, therefore, to consider the expression $\frac{dp}{\rho a}$ as a differential. Let

T temperature

i heat content (enthalpy)

s entropy

By the second main theorem

$$T\,ds = di - \frac{1}{\rho}\,dp$$

From this, on account of the hypothesis of constant entropy,

$$\frac{1}{\rho}\left(\frac{dp}{dT}\right)_s = \left(\frac{di}{dT}\right)_s$$

With that, it follows that

$$\frac{dp}{\rho a} = \frac{1}{a}\left(\frac{di}{dT}\right)_s \, dT$$

$$W(T) = \frac{1}{a_o} \int_{T_o}^{T} \frac{1}{a} \left(\frac{di}{dT}\right)_s dT \tag{17}$$

is introduced in which a_o is the sonic velocity of a comparison phase which was added to make W dimensionless. The phase of the gas may be characterized by W from $W = W(T)$. It follows that

$$T = T(W) \tag{18a}$$

Further, it is valid that

$$p = p(T) = p(W) \tag{18b}$$

$$a = a(T) = a(W) \quad \text{etc.} \tag{18c}$$

With the use of W equations (14) appear in the form

$$a_o\, dW + dv = 0 \qquad \text{for a Mach wave 1}$$

$$a_o\, dW - dv = 0 \qquad \text{for a Mach wave 2}$$

Bringing in

$$\lambda = W + \frac{v}{a_o} \tag{19a}$$

$$\mu = W - \frac{v}{a_o} \tag{19b}$$

these last relations change to a form which may be integrated. This gives

$$\lambda = \text{constant} \qquad \text{for Mach wave 1} \tag{20a}$$

$$\mu = \text{constant} \qquad \text{for Mach wave 2} \tag{20b}$$

If the magnitudes of λ and μ are known for a point of the yt-diagram, the velocity is thereby completely defined as well as the thermodynamic phase. It is, to be exact,

$$W = \frac{\lambda + \mu}{2} \tag{21a}$$

$$\frac{v}{a_o} = \frac{\lambda - \mu}{2} \tag{21b}$$

and on account of equations (18)

$$p = p(\lambda + \mu) \tag{22a}$$

$$a = a(\lambda + \mu) \tag{22b}$$

The next sections explain this transformation and the application of equations (20) to an example of an ideal gas whose specific heat is a function of temperature.

7. THERMODYNAMIC RELATIONS FOR AN IDEAL GAS WHOSE SPECIFIC HEAT IS A FUNCTION OF TEMPERATURE; COMPUTATION OF W

Let

c_p specific heat at constant pressure

c_v specific heat at constant volume

R gas constant

For an ideal gas

$$\frac{p}{\rho} = RT \tag{23}$$

According to the second main theorem, if p and T are considered as independent variables

$$ds = \frac{1}{T}\frac{\partial i}{\partial T}\,dT + \frac{1}{T}\frac{\partial i}{\partial p}\,dp - \frac{1}{\rho T}\,dp \tag{24}$$

Since ds is a perfect differential,

$$\frac{\partial}{\partial p}\left(\frac{1}{T}\frac{\partial i}{\partial T}\right) = \frac{\partial}{\partial T}\left(\frac{1}{T}\frac{\partial i}{\partial p} - \frac{1}{\rho T}\right)$$

Accordingly, substituting ρ from (23), the following known fact is obtained

$$\frac{\partial i}{\partial p} = 0$$

that is

$$i = i(T) \tag{25}$$

now

$$\left(\frac{\partial i}{\partial T}\right)_p = c_p \tag{26}$$

therefore

$$c_p = c_p(T)$$

From (24) as a result

$$ds = \frac{c_p}{T}\, dT - R\, \frac{dp}{p} \tag{24a}$$

and from this by integration

$$\frac{s - s_o}{R} = \int_{T_o}^{T} \frac{c_p}{R} \frac{dT}{T} - \ln \frac{p}{p_o}$$

the index o characterizes a comparison phase.

Introducing

$$P = e^{\int_{T_o}^{T} \frac{c_p}{R} \frac{dT}{T}} \tag{27a}$$

$$\pi = e^{\frac{-s-s_o}{R}} \tag{27b}$$

gives

$$\frac{p}{p_o} = P\pi \tag{28}$$

Considering $c_p(T)$ as known, p by (28) and ρ by (23), are given as functions of T and s; the thermodynamic properties of the medium can be calculated in principal, therefore. The quantity a is also defined by p and ρ. The computation of a is by all means simpler if carried out in the following way. According to (5)

$$\frac{1}{a^2} = \frac{\partial \rho}{\partial p}$$

for which the entropy is to be kept constant. For constant entropy from (24a)

$$\frac{dT}{T} = \frac{R}{c_p}\frac{dp}{p}$$

by differentiation from (23)

$$\frac{dp}{p} - \frac{d\rho}{\rho} = \frac{dT}{T}$$

From the last two relations together with the familiar relation

$$c_p - c_v = R$$

is obtained

$$a(T) = \sqrt{\frac{c_p}{c_v} RT} \tag{29}$$

The relations discovered up until now describe the properties of the gas and must always be known; it makes no difference which variation of the characteristic method is chosen for the calculation of the flow. In contrast, the introduction of the functions W, λ, and μ serve only as preparation for carrying out of the characteristics method in the form presented in the preceding section. Next, to compute W. From (25) it follows

$$\left(\frac{\partial i}{\partial T}\right)_p = \left(\frac{\partial i}{\partial T}\right)_s$$

with (26)

$$\left(\frac{\partial i}{\partial T}\right)_s = c_p$$

Setting the last equation as well as (29) into (17) gives

$$W = \frac{\lambda + \mu}{2} = \frac{1}{a_o}\int_{T_o}^{T}\sqrt{\frac{c_p c_v}{RT}}\,dT \tag{30}$$

For the present case, s = constant = s_o (28) becomes

$$\frac{p}{p_o}(T) = e^{\int_{T_o}^{T}\frac{c_p}{R}\frac{dT}{T}} \tag{28a}$$

Now the following can be formed

$$T = T(\lambda + \mu)$$

$$a = a(\lambda + \mu)$$

$$P = P(\lambda + \mu)$$

These calculations were carried out numerically for carbon dioxide. The relation between the specific heat and temperature was taken from Hütte[5] with the aid of these values $(i - i_o)/a_o^2$, a/a_o, P, and W can be computed from equations (26), (29), (27a), and (30) as functions of the temperature. (See figs. 2(a) and 2(b).) Figure 3 shows a/a_o, P, and T plotted as functions of $\lambda + \mu = 2W$.

8. THE CONSTRUCTION OF THE FLOW FIELD

The following problem should be dealt with: Along a curve K of the yt-diagram, which has at the most one point in common with each characteristic, let p/p_o and v/a_o be given (fig. 4). The flow should be constructed for the following times as far as it is defined by the portion of K given. Therefore, it is concerned here with the computation of the part of the flow defined by the initial conditions which by the same arguments appear everywhere in the interior, too. Before the construction of the flow can be started

[5]Hütte, 27th edition, Vol. 1, p. 48, table 5, Berlin 1941, Wilhelm Ernst und Sohn, publishers.

the initial values p/p_o and v/a_o must be expressed in terms of the variables λ and μ. Since the entropy was assumed constant, p/p_o and P coincide. With the aid of the relation presented in figure 3, between P and $\lambda + \mu$ and equation (21b), λ and μ may be ascertained without difficulties. Figure 4 shows on the right, the yt-diagram, on the left, the diagram of the assumed values p/p_o and v/a_o as well as those of the computed quantities λ and μ as functions of y.

Proceeding from the individual points of K if the network of Mach waves had been brought in, the phase at each lattice point would be determined thereby; according to (20) λ is constant along Mach wave 1, μ along Mach wave 2, and on that account, equal to the values at those points of K from which the Mach waves spread out. By (21b) and (22) the phase is given by λ and μ. To be able to draw the network of Mach waves, only their directions are still needed. These are given at the lattice points by (10); a/a_o is a function of $\lambda + \mu$ in figure 3, v/a_o is computed as $\frac{\lambda - \mu}{2}$.

The direction for the portion of a Mach wave between two lattice points is approximated as the average value of the corresponding directions at the lattice points.

The construction becomes especially simple if the Mach waves are drawn for equidistant values of λ and μ. The directions of the Mach waves appearing can be computed beforehand and possibly prepared in the form of table I. The interval between adjacent values of λ or μ was selected as 0.1, the size of the interval depends on the accuracy desired. In the table the upper column headings and signs refer to Mach wave 1, the lower to Mach wave 2. The numbers entered in the table represent the average values for $(v + a)/a_o$ and $(v - a)/a_o$. For Mach wave 1 for which $\lambda = 0.3$ and which leads from a point with $\mu = 0.2$ to a point with $\mu = 0.1$, in the column with the heading $\lambda = 0.3$ the value is to be taken from the row $\mu = 1.5$, that is, $(v + a)/a_o = 1.103$.

In the flow diagram the values of λ valid there are entered to the left of the lattice point and the values of μ to the right. To determine, for example, the position of C from the points A and B since the phase of C is given beforehand by $\lambda = 1.1$ and $\mu = 0.5$ the average directions of the Mach waves $(v + a)/a_o = 1.422$, $(v - a)/a_o = -0.778$ can be taken from table I and drawn in the yt-diagram. The auxiliary diagram on the left in figure 4 can be used for this. There the direction of a Mach wave for which $(v + a)/a_o = 0.8$ is drawn in. Similar diagrams can be used as aids for the following examples, too. The portions of the Mach wave going out from K really require a special computation since the average values of λ or μ for them do not agree, in general, with the values of table I. The small deviation was tolerable, however.

9. FLOWS WITH CONSTANT ENTROPY IN A PIPE OF VARIABLE CROSS SECTION

If the cross section of the pipe is not constant, the right side of equations (14) from which it is necessary to start out, here too, are preserved. With that, there is the possibility of undertaking that integration along the Mach waves which led to equations (20). Nevertheless, the introduction of λ and μ still remains useful. Setting

$$\frac{a}{a_o}\left(\frac{v}{a_o}\frac{\partial \ln F}{\partial y} + \frac{1}{a_o}\frac{\partial \ln F}{\partial t}\right) = M \tag{31}$$

then

$$\frac{d\lambda}{dt} = -a_o M \tag{32a}$$

is obtained as the consistency condition for Mach wave 1 and

$$\frac{d\mu}{dt} = -a_o M \tag{32b}$$

for Mach wave 2.

The consistency conditions in the form of (32) contain at any given time the differential of only one of the unknown quantities λ or μ while the differentials of both p and v appear in (14) already. This implies an appreciable improvement in the numerical calculation.

The construction of the flow rests on the fact that equations (32) are considered different equations. Let G_A be the value which a quantity G assumes at the point A, ΔG_{BA} the difference $G_B - G_A$ and G_{mBA} an average value of G taken between A and B.

Applying equations (32) to compute from two known points A and B the phase at a third, C, which is on the same Mach wave, in the form

$$\left.\begin{aligned} \lambda_C - \lambda_A = \Delta\lambda_{C,A} = -M_{mAC}a_o\ \Delta t_{CA} \\ \mu_C - \mu_B = \Delta\lambda_{C,B} = -M_{mBC}a_o\ \Delta t_{CB} \end{aligned}\right\} \qquad (33)$$

For the determination of the flow λ_C and μ_C have to be computed by means of these last equations and, at the same time, the position ascertained of the points sought in the yt-diagram by the use of equations (10). The calculation process might be explained by an example.

The flow is considered as given along a curve of the yt-diagram and, admittedly by λ and μ (fig. 5, table II). In addition, the pipe cross section must be a known function of y and t. For that it is only necessary to require that F can be differentiated with respect to position and time, a premise which is always fulfilled in practice. For this example F is taken in the form

$$F = F_o y^2 t$$

From (31) for M

$$M = \frac{a}{a_o}\left(\frac{v}{a_o}\frac{2}{y} + \frac{1}{a_o t}\right)$$

The positions $y = 0$ and $t = 0$ for which M goes to infinity do not belong to this region of flow where such singularities appear (for example at the center of spherical waves); it is necessary to make special investigations which cannot be entered into in the present report[7].

The best way to follow the calculation is by means of the systematic calculation in table II. To facilitate comparison with the description the columns are numbered. The first column contains the designation of the point which is to be computed, the second column gives the known point which, in common with the point to be computed, is on Mach wave 1. Column 3 contains the corresponding

[7]Compare G. Guderley. "Starke kugelige oder zylindrische Verdichtungsstösse in der Nähe des Kugelmittelpunktes oder der Zylinderachse." Luftfahrtforschung, Bd. 19 (1942), pp.302-312. This concerns itself with a complicated special case of such a singularity.

point for Mach wave 2. The first five rows reproduce the initial values as well as some further values that hold at the given points which are necessary for a later calculation. The calculation of a new point is carried out in the form of an iteration method; as an example the point 4 will be explained. Next, the values for λ_4 and μ_4 are estimated (columns 4 and 5). In order not to use too favorable an estimate, it is assumed that $\lambda_4 = \lambda_1$ and $\mu_4 = \mu_2$. The quantity $(\lambda + \mu)_4$ is determined for these magnitudes and from that, with the aid of figure 3 $\left(\frac{a}{a_o}\right)_4$ and, farther on $\left(\frac{v}{a_o}\right)_4$ (columns 6 and 8). With these values $\left(\frac{v + a}{a_o}\right)_4$ and $\left(\frac{v - a}{a_o}\right)_4$ are computed (columns 9 and 10). Now the average directions for Mach waves 1 and 2 $\left(\frac{v + a}{a_o}\right)_{m1,4}$ and $\left(\frac{v - a}{a_o}\right)_{m2,4}$ are formed (columns 14 and 19) and the Mach waves are plotted on the yt-diagram. From this y_4 and a_ot_4 (columns 12 and 13) are obtained. With these values M_4 (column 11) and the average values $M_{m1,4}$ and $M_{m2,4}$ (columns 15 and 20) are computed. To continue for Mach waves 1 and 2 $\Delta a_ot_{4,1} = a_ot_4 - a_ot_1$ and $\Delta a_ot_{4,2} = a_ot_4 - a_ot_2$ have to be computed (columns 16 and 21) and can be substituted in equations (33). The quantities $\Delta\lambda_{4,1}$ and $\Delta\mu_{4,2}$ as well as λ_4 and μ_4 (columns 17, 18, 22, 23) are obtained. If the values λ and μ calculated in this manner do not agree well enough with the original estimate, the calculation must be repeated in which λ and μ just calculated appear in place of the earlier estimates. Naturally, the Mach waves must be plotted over again, too, in the yt-diagram for this. These figures only show the final form at any instant. For that reason all the steps in the iteration method are put in the tables. A good view of the results of the calculation as well as insight into the estimates to be carried out by the iteration method is obtained, if the flow is followed simultaneously in a $\lambda\mu$-diagram, as well as the yt-diagram (fig. 5, right). There the λ-axis was selected slanting up to the right at 45° and the μ-axis downward at 45°. With a suitable vertical scale $\lambda - \mu$, and therefore v/a_o, is obtained immediately on a horizontal scale $\lambda + \mu$ or W and with the use of unequal distributions a/a_o and P and, for isentropic flows p/p_o too. The λ- and μ-axes were inclined 45° to obtain the quantities of physical interest v/a_o, a/a_o, etc. in a coordinate system with the conventional arrangement.

10. FLOW OF AN IDEAL GAS WITH ENTROPY DIFFERENCES

The introduction of λ and μ with the object of obtaining equations in only one unknown, at any time, with the iteration method for the determination of the flow was possible up until now because the expression $\frac{dp}{\rho a}$ with constant entropy might have been considered as the differential of a function W independently of the characteristics of the incident gas. Naturally, that is no longer possible with variable entropy. The computation of the flow must, in general, therefore, return to (14). The ideal gases constitute an exception. Here, as recognized in (30), the function W which essentially agrees with $\int \frac{dp}{\rho a}$ for constant entropy, depends on the temperature alone, and no longer on the entropy. If the expression $\frac{dp}{\rho a}$ is considered, therefore, in the case of variable entropy as dependent on the variables T and s the effect of change in entropy is separated, then the rest can be written here as a differential and λ and μ can be introduced as previously. The change in the entropy along the Mach waves must naturally be regarded separately. This is possible without especial difficulties since the entropy is constant along the life lines. The transformations are carried through in the following manner. From the second law

$$T\,ds = di - \frac{1}{\rho}\,dp$$

taking into account (26) and (29)

$$\frac{1}{\rho a}\,dp = \frac{di}{a} - T\,\frac{ds}{a} = \sqrt{\frac{c_p c_v}{RT}}\,dT - \sqrt{\frac{c_v}{c_p}\frac{T}{R}}\,ds$$

Introducing W, λ, and μ as before, the consistency conditions are obtained in the form

$$\frac{d\lambda}{dt} = -\frac{a}{a_o}\left(v\,\frac{\partial \ln F}{\partial y} + \frac{\partial \ln F}{\partial t}\right) + \sqrt{\frac{c_v}{c_p}\frac{T}{R}}\,\frac{1}{a_o}\,\frac{ds}{dt} \qquad \text{(34a) for Mach wave 1}$$

$$\frac{d\mu}{dt} = -\frac{a}{a_o}\left(v\,\frac{\partial \ln F}{\partial y} + \frac{\partial \ln F}{\partial t}\right) + \sqrt{\frac{c_v}{c_p}\frac{T}{R}}\,\frac{1}{a_o}\,\frac{ds}{dt} \qquad \text{(34b) for Mach wave 2}$$

The differential quotients ds/dt formed along the Mach waves interfere: The following transformations are possible. Analogous to the flow function of two-dimensional stationary flows, a function ψ is introduced, ψ is constant along the life lines. This can be achieved by requiring that

$$\frac{\partial\psi}{\partial t} = -\frac{F}{F_o}\frac{\rho}{\rho_o} v \tag{35a}$$

$$\frac{\partial\psi}{\partial y} = \frac{F}{F_o}\frac{\rho}{\rho_o} \tag{35b}$$

Along any curve of the yt-diagram

$$\frac{d\psi}{dy} = \frac{\partial\psi}{\partial y} + \frac{\partial\psi}{\partial t}\frac{dt}{dy} \tag{36}$$

Along a life line $\frac{dy}{dt} = v$ therefore

$$\frac{d\psi}{dy} = \frac{F}{F_o}\frac{\rho}{\rho_o} - \frac{F}{F_o}\frac{\rho}{\rho_o}\frac{v}{v} = 0$$

that is ψ is actually, constant along the life line. At each point of the yt-diagram ψ itself can be defined by a line integral that leads from a fixed point A at which ψ might be zero to B.

$$\psi_B = \int_A^B\left(\frac{\partial\psi}{\partial y}\,dy + \frac{\partial\psi}{\partial t}\,dt\right) = \int_A^B\left(\frac{F}{F_o}\frac{\rho}{\rho_o}\,dy - \frac{F}{F_o}\frac{\rho}{\rho_o}v\,dt\right) \tag{37}$$

The physical significance of ψ can be recognized as follows: Let C (fig. 6) be the intersection point of the life line through A with the line t = constant through B. To begin with, the path of integration is along the life line from A to C and, from there, out along the line t = constant to B. Along the life line AC, ψ is constant

$$\psi_C = \psi_A = 0$$

along the section CB, dt is equal to zero, accordingly

$$\psi_B = \int_C^B \frac{F}{F_o} \frac{\rho}{\rho_o} dy$$

From this, it is evident that ψ represents the mass which is enclosed between the particles at an instant in time for which ψ is zero.

The fact that s is constant along a life line can be written with the use of ψ in the form

$$s = s(\psi) \tag{38}$$

For ds/dt then

$$\frac{ds}{dt} = \frac{ds}{d\psi} \frac{d\psi}{dt}$$

for which $\frac{dy}{dt}$ is to be taken, just as ds/dt previously, along the Mach wave considered.

From (35) and (10)

$$\frac{d\psi}{dt} = \frac{F}{F_o} \frac{\rho}{\rho_o} a \qquad \text{for Mach wave 1}$$

$$\frac{d\psi}{dt} = -\frac{F}{F_o} \frac{\rho}{\rho_o} a \qquad \text{for Mach wave 2}$$

Substituting these in equations (34), allowing for (23), (28), and (29) replacing $\frac{ds/R}{d\psi}$ according to (27b) by $-\frac{1}{\pi} \frac{d\pi}{d\psi}$ and p_o/ρ_o by $\frac{C_{v_o}}{C_{p_o}} a_o^2$ yields the following consistency conditions:

For Mach wave 1

$$\frac{d\lambda}{dt} = a_o\left[-\frac{a}{a_o}\left(\frac{v}{a_o}\frac{\partial \ln F}{\partial y} + \frac{1}{a_o}\frac{\partial \ln F}{\partial t}\right) - \frac{C_{v_o}}{C_{p_o}}\frac{F}{F_o}P\frac{d\pi}{d\psi}\right] \tag{39a}$$

For Mach wave 2

$$\frac{d\mu}{dt} = a_o\left[-\frac{a}{a_o}\left(\frac{v}{a_o}\frac{\partial \ln F}{\partial y} + \frac{1}{a_o}\frac{\partial \ln F}{\partial t}\right) + \frac{C_{v_o}}{C_{p_o}}\frac{F}{F_o}P\frac{d\pi}{d\psi}\right] \tag{39b}$$

Here P is a function of $\lambda + \mu$ (fig. 3), F/F_o is known to be a function of y and t. From (38) and (27b) it follows that

$$\pi = \pi(\psi)$$

and from this

$$\frac{d\pi}{d\psi} = \frac{d\pi}{d\psi}(\psi) = \text{constant for a life line} \tag{40}$$

For the sake of compactness, introducing

$$N = \frac{C_{v_o}}{C_{p_o}}\frac{F}{F_o}P\frac{d\pi}{d\psi}$$

Then (39) goes over into the form

$$\frac{d\lambda}{dt} = a_o(-M + N) \qquad \text{(41a) for Mach wave 1}$$

$$\frac{d\mu}{dt} = a_o(-M - N) \qquad \text{(41b) for Mach wave 2}$$

Equations (40) and (41) supplant the previous consistency conditions (15) and (14).

Before starting the characteristic construction, the problem arises here, too, of computing λ and μ, and now $\frac{d\pi}{d\psi}$ besides, from the initial values. Along a curve K of the yt-diagram let the velocity be given by v/a_o, the phase of the gas by p/p_o and T. From T with the aid of figure 3 $\lambda + \mu$ is obtained, from v/a_o, $\lambda - \mu$; with this λ and μ are known. Since p/p_o are given, and P as a function of T is to be gathered from figure 2, π is obtained immediately from (28). As a result of plotting π against the values of y from the curve K and differentiating $\frac{d\pi}{dy}$ is obtained. From (36) and (35) together with (23) $\frac{d\psi}{dy}$ for the curve K may be computed for the curve K and, finally, with that

$$\frac{d\pi}{d\psi} = \frac{d\pi}{dy}\frac{d\psi}{dy}$$

is determined. In many cases these computations are superfluous; if entropy differences arise from compression shocks, the determination of $\frac{d\pi}{d\psi}$, λ and μ includes their calculation. The way the computation of flow has to be carried out is shown in figure 7 and table III with points 4, 5, 6 as examples. The related $\lambda\mu$-diagram is right center. (The points included, in addition, in the table and the figures relate to a later section.) Along the curve K (points 1-3) λ, μ, and $\frac{d\pi}{d\psi}$ are assumed as known, in the auxiliary diagram $\frac{d\pi}{d\psi}$ has been reproduced as a function of y. The computation of a new point - take point 4 as an example - begins, here too, with an estimate of λ and μ (table III, columns 4 and 5). After that, as before, the following are computed $(\lambda + \mu)_4$; $(a/a_o)_4$; $(v/a_o)_4$; $\left(\frac{v + a}{a_o}\right)_4$; $\left(\frac{v + a}{a_o}\right)_4$; $\left(\frac{v + a}{a_o}\right)_{m1,4}$; $\left(\frac{v - a}{a_o}\right)_{m2,4}$; (columns 6-10, 19 and 24), the position of 4 is indicated in the yt-diagram and y_4 and $a_o t_4$ in the table (columns 11 and 12) assumed. The determination of $\frac{d\pi}{d\psi}$ with the aid of the life lines enters in as something new. It should be sufficient for this to draw in a multitude of life lines, simultaneous with the construction of the Mach waves and going back over these to learn the desired value $\frac{d\pi}{d\psi}$ from the auxiliary diagram. The $\lambda\mu$-diagram is useful for a quick determination of the direction of the life lines. The position of the intersection points of the life lines with the Mach waves may be estimated there without difficulty, and then the average velocity learned. (Compare points 14 and 15 in the yt- and in the $\lambda\mu$-diagram.) After $\frac{d\pi}{d\psi}$ has been found and, in addition, P has been learned from diagram 3 (columns 13 and 14), M_4 and N_4 as well as $(-M - N)_4$ and $(-M + N)_4$ may be computed (columns 15-18), the average values $(-M - N)_{m1,4}$ and $(-M + N)_{m2,4}$ for the Mach waves be formed (columns 20 and 25) and with the aid of $\Delta a_o t$ (columns 21 and 26) from equations (41) compute $\Delta\lambda$ and $\Delta\mu$ and, ultimately with that λ and μ. (Columns 22, 23, 27, and 28.) Where the original estimates were too bad, the computation was repeated.

11. THE GENERALIZED FORM OF THE CHARACTERISTICS METHOD

An outline shall be given of how to proceed if the simplifications given above are no longer possible or if the flow is so small that the prepared computations as given at the end of section 7 do not pay. As an example, let the computation of the point 4 be carried through from the points 1 and 2 of figure 7. (See fig. 8.) The quantities

$$p_1/p_o = 1.44; \qquad \pi_1 = 1.2; \qquad v_1/a_o = 0.425$$

$$p_2/p_o = 1.866; \qquad \pi_2 = 1.332; \qquad v_2/a_o = 0.400$$

correspond to the initial values assumed there. For the medium to be investigated ρ and a must be given as functions of p and π. In this case P is obtained, first of all, from (28) and from that and figure 2(b), T. Then ρ/ρ_o and a/a_o are obtained with (23) and (29). Hence

$$a_1/a_o = 1.021; \qquad \rho_1/\rho_o = 1.375; \left(\frac{v + a}{a_o}\right)_1 = 1.446$$

$$a_2/a_o = 1.037; \qquad \rho_2/\rho_o = 1.710; \left(\frac{v - a}{a_o}\right)_2 = 0.637$$

besides

$$M_1 = 1.747; \quad M_2 = 1.442$$

can be computed. Here, too, an estimate is made in computing a new point. For example

$$p_4/p_o = p_1/p_o = 1.44; \quad v_4/a_o = v_1/a_o = 0.425; \quad \pi_4 = \frac{\pi_1 + \pi_2}{2} = 1.266$$

With this $P_4 = \left(p_4/p_o\right)/\pi_4 = 1.137$ is obtained, whence

$$T_4 = 282.5$$

Continuing further

$$a_4/a_o = 1.014; \quad \rho_4/\rho_o = 1.390; \quad \left(v_4 + a_4\right)/a_o = 1.439$$

$$\left(v_4 - a_4\right)/a_o = -0.589; \quad (v + a)_{m1,4}/a_o = 1.443; \quad (v - a)_{m2,4}/a_o = -0.613$$

With that the position of point 4 in the yt-diagram may be found, giving

$$y_4 = 1.446; \quad a_o t_4 = 1.258; \quad \Delta(a_o t)_{4,1} = 0.048; \; \Delta(a_o t)_{4,2} = 0.092$$

and, after further calculation

$$M_4 = 1.403$$

The average values are found to be

$$\left(\rho/\rho_o\right)_{m1,4} = 1.383; \quad \left(a/a_o\right)_{m1,4} = 1.0175; \quad M_{m1,4} = 1.439$$

$$\left(\rho/\rho_o\right)_{m2,4} = 1.550; \quad \left(a/a_o\right)_{m2,4} = 1.026; \quad M_{m2,4} = 1.423$$

Considering equations (14) as difference equations then[7]

$$\frac{p_o}{\rho_o a_o}\left(\frac{\rho_o}{\rho}\right)_{m1,4}\left(\frac{a_o}{a}\right)_{m1,4}\left(\frac{p_4}{p_o} - \frac{p_1}{p_o}\right) + a_o\left(\frac{v_4}{a_o} - \frac{v_1}{a_o}\right) = -a_o\left(a_o t_4 - a_o t_1\right)M_{m1,4}$$

$$\frac{p_o}{\rho_o a_o}\left(\frac{\rho_o}{\rho}\right)_{m2,4}\left(\frac{a_o}{a}\right)_{m2,4}\left(\frac{p_4}{p_o} - \frac{p_2}{p_o}\right) - a_o\left(\frac{v_4}{a_o} - \frac{v_2}{a_o}\right) = -a_o\left(a_o t_4 - a_o t_1\right)M_{m2,4}$$

Replacing $\frac{p_o}{\rho_o}$ by $\frac{C_{v_o}}{C_{p_o}}a_o^2$ from (29) and (23) gives

$$\frac{C_{v_o}}{C_{p_o}}\left(\frac{\rho_o}{\rho}\right)_{m1,4}\left(\frac{a_o}{a}\right)_{m1,4}\frac{p_4}{p_o} + \frac{v_4}{a_o} = \frac{C_{v_o}}{C_{p_o}}\left(\frac{\rho_o}{\rho}\right)_{m1,4}\left(\frac{a_o}{a}\right)_{m1,4}\frac{p_1}{p_o}$$
$$+ \frac{v_1}{a_o} - M_{m1,4}\left(a_o t_4 - a_o t_1\right)$$

$$\frac{C_{v_o}}{C_{p_o}}\left(\frac{\rho_o}{\rho}\right)_{m2,4}\left(\frac{a_o}{a}\right)_{m2,4}\frac{p_4}{p_o} - \frac{v_4}{a_o} = \frac{C_{v_o}}{C_{p_o}}\left(\frac{\rho_o}{\rho}\right)_{m2,4}\left(\frac{a_o}{a}\right)_{m2,4}\frac{p_2}{p_o}$$
$$- \frac{v_2}{a_o} - M_{m2,4}\left(a_o t_4 - a_o t_2\right)$$

---

[7]For ideal gases the first term of the left side of (14) may be written $\frac{1}{k}\frac{d\ln p}{dt}a$, then ρ/ρ_o does not have to be computed separately. To permit the procedure to be applicable in more general cases, this simplification is not used here.

Putting in numerical values gives as a result

$$0.547\ p_4/p_o + v_4/a_o = 1.1430$$

$$0.484\ p_4/p_o - v_4/a_o = 0.3715$$

$$p_4/p_o = 1.470$$

$$v_4/a_o = 0.342$$

From the velocity computed above v_4/a_o and the velocity at a point 4', estimated for the present, of the connecting line 1.2, the average direction of the life line passing through 4 is obtained by an approximation method. If this is proceeding from 4 backwards, the more accurate position of 4' is obtained. By interpolation between 1 and 2 $\pi_4{}' = \pi_4 = 1.243$ is obtained. Since the values $\frac{p_4}{p_o}$, $\frac{v_4}{a_o}$, π_4 do not agree sufficiently well yet with the originally estimated values, the computation must be repeated with the magnitudes just obtained as starting values. This gives

$$p_4/p_o = 1.478; \quad v_4/a_o = 0.3383; \quad \pi_4 = 1.243$$

12. SIMPLIFICATIONS FOR IDEAL GASES WITH CONSTANT SPECIFIC HEATS

Generally the flowing medium is an ideal gas with constant specific heat or at least can be considered as such, as an approximation. In such a case appreciable simplifications are possible. Let

$$k = c_p/c_v$$

then

$$c_p = \frac{k}{k - 1} R; \quad c_v = \frac{1}{k - 1} R$$

From equations (27a), (29), and (30)

$$P = \left(\frac{T}{T_o}\right)^{\frac{k}{k-1}}; \quad \frac{a}{a_o} = \sqrt{\frac{T}{T_o}}; \quad a_o = \sqrt{kRT_o}; \quad W = \frac{\lambda + \mu}{2} = \frac{2}{k - 1}\left(\frac{a}{a_o} - 1\right)$$

With this, it follows that

$$\lambda = \frac{2}{k - 1}\left(\frac{a}{a_o} - 1\right) + \frac{v}{a_o} \tag{42a}$$

$$\mu = \frac{2}{k - 1}\left(\frac{a}{a_o} - 1\right) - \frac{v}{a_o} \tag{42b}$$

and from that

$$\frac{a}{a_o} = 1 + \frac{k - 1}{4}(\lambda + \mu) \tag{42c}$$

consequently,

$$P = \left[1 + \frac{k - 1}{4}(\lambda + \mu)\right]^{\frac{2k}{k-1}} \tag{42d}$$

The directions of the characteristics are obtained from (9) and (10) in the form

$$\frac{dy}{dt} = a_o \frac{\lambda - \mu}{2} \qquad \text{for the life lines}$$

$$\frac{dy}{dt} = a_o\left(1 + \frac{k + 1}{4}\lambda - \frac{3 - k}{4}\mu\right) \text{ for Mach waves 1}$$

$$\frac{dy}{dt} = a_o\left(-1 - \frac{k + 1}{4}\mu + \frac{3 - k}{4}\lambda\right) \text{ for Mach waves 2}$$

The consistency conditions for the Mach waves remain unchanged in the form (40) and (41). M and N are expressed as follows, now,

$$M = \left[1 + \frac{k-1}{4}(\lambda + \mu)\right]\left(\frac{\lambda - \mu}{2}\frac{\partial \ln F}{\partial y} + \frac{1}{a_o}\frac{\partial \ln F}{\partial t}\right)$$

$$N = \frac{1}{4}\frac{F}{F_o}\left[1 + \frac{k-1}{4}(\lambda + \mu)\right]^{\frac{2k}{k-1}}\frac{d\pi}{d\psi}$$

The directions of the characteristics may now be found very conveniently graphically. A construction which is suitable if the simultaneous treatment of the flow in a $\lambda\mu$-diagram is avoided is the contribution of Adam Schmidt. (See fig. 9.) For the determination of the direction dy/dt for a life line, two vertical scales at a distance of 1 apart are used with $\frac{\lambda}{2}$ plotted on the right one and $\frac{\mu}{2}$ on the left one as above. A life line for a phase which is given by λ and μ has the direction of the connecting line of the points concerned on the function scales. Similarly, there are scales to use for a Mach wave 1, which give $\frac{3-k}{4}\mu$ on the left and $1 + \frac{k+1}{4}\lambda$ on the right. For Mach wave 2 $\frac{k+1}{4}\mu$ has been plotted on the left and $-1 + \frac{3-k}{4}\lambda$ on the right. In figure 9, the direction of Mach waves 1 and the life line is given for $\lambda = 1.1$ and $\mu = 0.6$.

If the phases in the course of the construction of a $\lambda\mu$-diagram are followed up, the following method is suitable (fig. 10, right). A vertical line is sent through the 0-point of the $\lambda\mu$-system and the poles P_1, P_2, and P_L are determined, where P_L is on a level with the origin of the $\lambda\mu$-system and $\sqrt{2}$ away from it. P_1 and P_2 are directly below and above P_L, respectively, and likewise the distance $\sqrt{2}$ from it. To find the direction of the characteristics for a given phase, a horizontal ray and two rays slanting upward and downward at an angle arc tan $\frac{k-1}{2}$ are drawn. These intersect the vertical line through the origin of the $\lambda\mu$-system at the points Q_1, Q_2, and Q_L. The connecting lines P_1Q_1, P_2Q_2, and P_LQ_L are the directions of Mach waves 1 and 2 and the life line. In figure 10 the construction for point 4 is carried out.

This construction is especially convenient with a triangle having an angle arc tan $\frac{k-1}{2}$. Figure 10 and table 4 give an example of an application for the same initial values as in figure 7 and with c_p/c_v = constant = 1.4.

13. BOUNDARY CONDITIONS

If the flowing gas column is not infinite, the variation of the flow is determined by the phase at the start, in addition, also by conditions at its boundaries. For example, a gas can be closed off by a piston or rigid wall, flow out into a space with a given pressure, or be sucked out of the same. Generally, the boundary conditions may be formulated so that relations between the phase magnitudes of the gas and its velocity along a curve of the yt-diagram are prescribed. The number of conditions which are needed for the boundary curve corresponds to the number of characteristics which run out from there into the interior of the flow. For example, the gas flows out of the end of the pipe into a space with constant pressure, with $v < a$, then the line y = constant is the curve for the pipe for which the boundary conditions are given. A family of Mach waves spreads out from it inward, while the other family and the life lines reach this curve, approaching it from within. In this case the condition can be prescribed that the pressure in the exit section be equal to the outside pressure. If the gas is sucked in from outside, Mach waves of the one family proceed from the curve of the boundary conditions as well as the life lines. Accordingly, two conditions must be given. The one states that the entropy of the entering particle is the same as the entropy in the outer space, as a second it would be required perhaps that the phase of the gas in the entrance section be related to the phase in the outer space through Bernoulli's equation[8]. (An exact formulation is difficult, since the flow at this location is no longer one-dimensional.) If the characteristics of all three families of a given curve lead out into the interior of the region to be computed, there are three conditions to prescribe; this is the initial value problem already treated. The other extreme, that at the boundary of the region of interest, generally, no condition can be fulfilled, is physically conceivable, too. For example, if a gas with $v > a$ flows in a space at constant pressure, generally no characteristic goes inward from the outflow section.

[8]Compare Schultz-Grunow, loc. cit.

Actually here – disregarding boundary conditions which force compression shocks – no effect on the flow variation in the interior is possible from outside.

The treatment of boundary conditions is explained with two examples which are connected with the flow in figure 7. The computation is entered in table III, as far as possible. The first example includes points 7 to 9 and, admittedly, it has been assumed that the gas column is bounded by a piston whose life line is represented in the yt-diagram as the curve 3, 7, 9. (Whether it is practicable to realize such a piston in a tube of variable cross section is unimportant for carrying out the computation.) The λμ-diagram referred to is in figure 7, upper right. To begin with, an estimate is made of the phase at 7 which has been chosen $\lambda_7 = \lambda_3 = 0.800$, $\mu_7 = \mu_3 = 0.050$. Since the line 3.7 is the life of a particle, $\left(\frac{d\pi}{d\psi}\right)_7$ is already known and is equal to $\left(\frac{d\pi}{d\psi}\right)_3$.

With this the values in columns 6–10 and 19 are calculated. As a result of drawing in the Mach wave 5.7, y_7 and $a_o t_7$ (columns 11 and 12) are obtained and besides v_7/a_o from the direction of the life line at point 7 which has been reached. (This quantity is found in column 8 under the value computed from the initial estimates.) Now the quantities in columns 14 to 18 and 20 to 23 may be computed, the value v_7/a_o obtained from the boundary conditions will be used. With that λ_7 is already known. The quantity μ_7 is obtained from the relation

$$\frac{v}{a_o} = \frac{\lambda - \mu}{2}$$

Inserting numbers

$$0.323 = 1/2. \; (0.417 - \mu_7); \quad \mu_7 = -0.229.$$

Since the first estimate was too poor, the computation must be repeated.

Point 8 is computed from 6 and 7 by the method explained in section 9. From 8, point 9 is obtained in the way just described.

This method of calculation is useful for any laws of motion of the pipe; a special argument is necessary only if a discontinuity appears. The discontinuity in the velocity is to be considered attained on transition of the boundary from a continuous velocity variation at very large acceleration. In the yt-diagram that means that the life line of the piston which has a bend at the instant of the velocity discontinuity is rounded off immediately. Then the flow may be drawn accurately just as previously. To obtain sufficient accuracy, enough points must be taken on the rounding off so that the velocity of the piston does not change excessively from point to point, and at each point a Mach wave of the first family may converge and a Mach wave of the second family may diverge from there. First of all λ must be computed for the converging Mach wave and then from λ and the velocity at the incident point μ determined for each Mach wave. If the rounding off becomes smaller and smaller, these points on the rounding off draw closer and closer. With that the values of λ approach a single value, which may be computed from the field before the bend. The Mach waves 2 spread out in the shape of a fan from the bend and the fan includes all values of μ which lie between the values of μ for the velocity before and after the velocity discontinuity.

For the second example, there is at the position $y = y_1$ an open pipe end, through which gas is sucked in from outside and for which two conditions must be specified along the boundary-condition curve. The curve is the curve 1, 10, 13 in figure 7. In the outer space let $\pi = \pi_1$; for the entering particle therefore $\frac{d\pi}{d\psi} = 0$. This is one boundary condition. As the second boundary condition there is the requirement that the phase in the inflow section be related to the phase in the outer space by the Bernoulli equation. This condition may be satisfied, already, at point 1, accordingly

$$i + v^2/2 = i_1 + v_1{}^2/2$$

or also

$$\frac{i - i_o}{a_o{}^2} + \frac{1}{2}\left(\frac{v}{a_o}\right)^2 = \frac{i_1 - i_o}{a_o{}^2} + \frac{1}{2}\left(\frac{v_1}{a_o}\right)^2 = \text{constant}$$

To determine these constants, from figure 3 the temperature T_1 is taken for $(\lambda + \mu)_1$ from figure 2(a) for T_1, $(i_1 - i_o)/a_o{}^2$.

Then

$$\frac{i_1 - i_o}{a_o{}^2} + \frac{1}{2}\left(\frac{v_1}{a_o}\right)^2 = 0.292$$

Since $(i - i_o)/a_o^2$ is a function of T and, therefore, of $\lambda + \mu$, $\frac{v}{a_o} = \frac{\lambda - \mu}{2}$ this boundary condition can be plotted as the curve K in the $\lambda\mu$-diagram (fig. 7, lower right). At best, the computation of point 10 begins anew with an estimate for λ and μ so that the boundary conditions are already satisfied (columns 4 and 5). With this, the quantities in columns 6, 7, 8, 10, and 24 are computed and Mach wave 2 drawn in with that. The quantity $a_o t_{10}$ is obtained in column 12, the values $y_{10} = y_1$ and $\frac{dr}{d\psi} = 0$. (Columns 12 and 13 are given beforehand.) Now the quantities in columns 14 to 18 can be obtained.

To determine, with this, the quantity $(-M + N)$ in column 25 it is to be noted that $(-M + N)$ for the particle originally in the pipe has the value, perhaps, at point 4 and changes discontinuously for the particle recently sucked into the quantity $(-M + N)_{10}$.

On that account the life line is drawn, which separates the particles in the interior originally from those particles flowing in from outside. This intersects Mach wave 4, 10 at point 11. Then the following is obtained (column 25)

$$(-M + N)_{m4,10} = \frac{1}{\Delta(a_o t)_{4,10}} \Big[\Delta(a_o t)_{4,11}(-M + N)_4 + \Delta(a_o t)_{11,10}(-M + N)_{10}\Big]$$

The quantities in columns 26, 27, and 28 may be computed now. As a result of inspecting the curve of the boundary condition in the $\lambda\mu$-diagram with the value of μ found, λ is obtained (column 23). The computation is repeated with the values found in this way.

From points 6 and 10, point 12 is obtained in the manner described in section 9. In connection with that the difficulty just described appears again in finding the average value for $(-M + N)$. From 12 and the boundary condition, point 13 may be computed by the method just presented.

The $\lambda\mu$-diagrams of the two last examples were kept separate from the $\lambda\mu$-diagram drawn for points 1-6 for the sake of clarity.

If the various figures are visualized as being joined - the upper diagram connected to the middle one at the line 6, 5, 3, the middle one with the lower one at the line 1, 4, 6 - it is recognized that the plane is covered with several sheets which are connected along the figures of the characteristics. There is such a superposition, already, in the lower $\lambda\mu$-diagram; there are to be imagined inclosed the quadrilateral 10, 4, 6, 12 along 10, 4, the triangle 1, 4, 10, along 10, 12 the triangle 10, 3, 12.

In addition to the boundary conditions, transitional conditions can also appear in the interior of the flow. In the example just discussed just that would have been the case, if in the outer space π were different from π_1. At the location of such a discontinuity for π agreement of pressure and velocity must be required. To go into such questions with greater detail lies beyond the scope of this report.

14. TRANSITIONAL CONDITIONS AT COMPRESSION SHOCKS

The flow in a given part of the yt-plane is defined by the initial and boundary conditions and is calculable by the methods derived up until now. It is possible that it might happen during the construction that regions of the yt-plane are covered with phase quantities several times. This is the sign for the appearance of compression shocks. The entropy is no longer constant after the passage of a compression shock. On that account the computation of compression shocks simultaneously includes the determination of the function $s(\psi)$ or $\frac{d\pi}{d\psi}(\psi)$, too, for the region of the yt-plane behind the compression shock.

For the mathematical treatment, a compression shock is to be considered a curve along which two flows collide, which are related to one another and to the direction of this curve by transition conditions. It will be the problem of this section to derive these (known of themselves)[9] transition conditions in a convenient form for the present purpose.

Proceeding from a stationary compression shock, that is from a compression shock whose front is at rest relative to the coordinate system selected, let the index I designate the phase before the

[9]Compare Ackeret for instance. Beitrag Gasdynamik in Handbuch der Physik, Bd. VII, p. 324 and following pages, Berlin 1927.

shock, the index II the phase after the shock. The additional index s might point out that this concerns the calculation of a stationary shock. Then the momentum and the energy theorems as well as the equation of continuity are written in the form

$$p_{Is} + \rho_{Is}v_{Is}^2 = p_{IIs} + \rho_{IIs}v_{IIs}^2 \tag{43a}$$

$$i_{Is} + \frac{1}{2}v_{Is}^2 = i_{IIs} + \frac{1}{2}v_{IIs}^2 \tag{43b}$$

$$\rho_{Is}v_{Is} = \rho_{IIs}v_{IIs} \tag{43c}$$

Furthermore, the characteristics of the gas concerned must be known, possibly in the form

$$p = p(i, \rho) \tag{43d}$$

If the quantities in advance of the shock i_{Is}, ρ_{Is}, and v_{Is} are known, then the compression shock is therewith calculable. Actually all three quantities enter into the general gas laws, too, as parameters. In order to carry out the computation practically, in such a case, ρ_{IIs} from (43c) and i_{IIs} from (43b) have to be expressed as functions of v_{IIs} and the known quantities and then substituted in (43a). With that, on account of (43d), p_{IIs}, too, is a function of v_{IIs} and the known quantities in advance of the shock. In this manner an equation for v_{IIs} alone is obtained which must be solved numerically in a suitable manner. For an ideal gas for which c_p is not constant, equations (43) transform with the aid of (23, as follows:

$$\frac{\rho_{Is}}{\rho_{IIs}} RT_{Is} + \frac{\rho_{Is}}{\rho_{IIs}} v_{Is}^2 = RT_{IIs} + v_{IIs}^2 \tag{44a}$$

$$i(T_{Is}) + \frac{1}{2}v_{Is}^2 = i(T_{IIs}) + \frac{1}{2}v_{IIs}^2 \tag{44b}$$

$$v_{Is} \frac{\rho_{Is}}{\rho_{IIs}} = v_{IIs} \tag{44c}$$

Since ρ_{Is} appears here only in the combination ρ_{IIs}/ρ_{Is} only T_{Is} and v_{Is} still remain as parameters upon which the phases behind the shock depend. To calculate the shock curves numerically, it is useful, first to regard T_{Is} and T_{IIs} as parameters and determine v_{Is} from this subsequently. The computation process is the following: From (44a) and (44c)

$$\frac{RT_I}{v_{Is}} + v_{Is} = \frac{RT_{II}}{v_{II}} + v_{IIs} \tag{45a}$$

As a result of squaring this

$$\frac{R^2 T_{Is}^2}{v_{Is}^2} + 2RT_{Is} + v_{Is}^2 = \frac{R^2 T_{IIs}^2}{v_{IIs}^2} + 2RT_{IIs} + v_{IIs}^2 \tag{45b}$$

Introducing

$$\Delta i = i_{IIs} - i_{Is} .$$

gives

$$v_{IIs}^2 = v_{Is}^2 - 2\Delta i \tag{46}$$

from (44b).

Putting this in (45b), the desired equation for v_{Is}^2 is obtained as

$$v_{Is}^4\left[2\Delta i - 2R\left(T_{IIs} - T_{Is}\right)\right] + v_{Is}^2\left[-4\Delta i^2 + 4R\Delta i\left(T_{IIs} - T_{Is}\right) - R^2\left(T_{IIs}^2 - T_{Is}^2\right)\right] - 2R^2 T_{Is}^2 \Delta i = 0$$

If v_{Is} is determined, then v_{IIs} and ρ_{IIs}/ρ_{Is} are computed in turn with the aid of (46) and (44c); finally

$$p_{IIs}/p_{Is} = \rho_{IIs}/\rho_{Is}\; T_{IIs}/T_{Is} .$$

For an ideal gas with constant specific heats, the following transformations may be undertaken. According to the familiar relations

$$i = \frac{k}{k-1} RT$$

and

$$a^2 = kRT$$

Equations (45a) and (44b) are written in the form

$$\frac{a_{Is}^2}{kv_{Is}} + v_{Is} = \frac{a_{IIs}^2}{kv_{IIs}} + v_{IIs}$$

$$\frac{2}{k-1} a_{Is}^2 + v_{Is}^2 = \frac{2}{k-1} a_{IIs}^2 + v_{IIs}^2$$

or

$$\frac{1}{k} \frac{1}{v_{Is}/a_{Is}} + \frac{v_{Is}}{a_{Is}} = \frac{1}{k}\left(\frac{a_{IIs}}{a_{Is}}\right)^2 \frac{1}{v_{IIs}/a_{Is}} + \frac{v_{IIs}}{a_{Is}} \tag{47a}$$

$$\frac{2}{k-1} + \left(\frac{v_{Is}}{a_{Is}}\right)^2 = \frac{2}{k-1}\left(\frac{a_{IIs}}{a_{Is}}\right)^2 + \left(\frac{v_{IIs}}{a_{Is}}\right)^2 \tag{47b}$$

By this, a_{IIs}/a_{Is} and v_{IIs}/a_{Is} and, with that, the other quantities, too, depend on the parameter v_{Is}/a_{Is} alone.

To compute v_{IIs}/a_{Is}, $(a_{IIs}/a_{Is})^2$ is eliminated:

$$\frac{k+1}{2}\left(\frac{v_{IIs}}{a_{Is}}\right)^2 - \frac{v_{IIs}}{a_{Is}}\left(k \frac{v_{Is}}{a_{Is}} + \frac{a_{Is}}{v_{Is}}\right) + \left(1 + \frac{k-1}{2} \frac{v_{Is}}{a_{Is}}\right)^2 = 0$$

is obtained as a result.

The solution of this equation is found, immediately, if it is borne in mind that on account of the form of (47) a solution is represented by

$$v_{IIs}/a_{Is} = v_{Is}/a_{Is}$$

then

$$\frac{v_{IIs}}{a_{Is}} = \frac{2}{k+1}\left(\frac{a_{Is}}{v_{Is}} + \frac{k-1}{2}\frac{v_{Is}}{a_{Is}}\right)$$

Using this, the following is obtained from (47b)

$$\left(\frac{a_{IIs}}{a_{Is}}\right)^2 = 1 + \frac{k-1}{2}\left[\left(\frac{v_{Is}}{a_{Is}}\right)^2 - \left(\frac{v_{IIs}}{a_{Is}}\right)^2\right]$$

and

$$\rho_{IIs}/\rho_{Is} = \left(v_{Is}/a_{Is}\right)\left(a_{Is}/v_{IIs}\right)$$

$$p_{IIs}/p_{Is} = \left(\rho_{IIs}/\rho_{Is}\right)\left(T_{IIs}/T_{Is}\right)$$

$$= v_{Is}/a_{Is}\ a_{Is}/v_{IIs}\left(a_{IIs}/a_{Is}\right)^2$$

The change of entropy is of interest, as well; with the aid of (27) and (28), these expressions result

$$\frac{s_{IIs} - s_{Is}}{R} = \frac{k}{k-1}\, ln\left(\frac{T_{IIs}}{T_{Is}}\right) - ln\frac{p_{IIs}}{p_{Is}}$$

$$\frac{s_{IIs} - s_{Is}}{R} = \frac{2k}{k-1}\, ln\frac{a_{IIs}}{a_{Is}} - ln\frac{v_{Is}}{a_{Is}} + ln\frac{v_{IIs}}{a_{Is}} - 2\,ln\frac{a_{IIs}}{a_{Is}}$$

$$= \frac{2}{k-1}\, ln\frac{a_{IIs}}{a_{Is}} - ln\frac{v_{Is}}{a_{Is}} + ln\frac{v_{IIs}}{a_{Is}}$$

From this

$$\frac{\pi_{IIs}}{\pi_{Is}} = \left(\frac{a_{IIs}}{a_{Is}}\right)^{\frac{2}{k-1}} \frac{v_{Is}}{a_{Is}} \frac{a_{Is}}{v_{IIs}}$$

Arbitrary compression shocks result from the stationary compression shocks just calculated because a velocity is superimposed. In doing so, the thermodynamic phase quantities before and after the shock for which accordingly the index s can be omitted are retained and moreover the velocity differences. Since the phase in advance of the shock is already given in the construction of flows, before the shock is computed, the relative velocities with respect to the phase in advance of the shock are formed. Let

u absolute velocity of shock front

Δu relative velocity of shock front with respect to particles in advance of shock

Then

$$\frac{\Delta u}{a_I} = \frac{-v_{Is}}{a_I}; \qquad \Delta v_{II,I} = v_{II} - v_I = -\frac{2}{k+1}\left(\frac{v_{Is}}{a_I} - \frac{a_I}{v_{Is}}\right)$$

The signs appearing in this are not astonishing. A stationary compression shock in a gas which moves in the positive direction propagates itself in a negative direction relative to the material ahead of the shock, and in so doing, produces a change in velocity in the direction of its propagation velocity, that is, in the negative direction, too. Naturally, compression shocks, which travel in the positive direction in the material at rest are also possible, the signs of the velocities have to be changed for these. The thermodynamic phase quantities of this are not touched upon. Corresponding to the distinction which had been met in Mach waves, these last compression shocks are designated compression shocks of the first type, those which propagate in the negative direction as compression shocks of the second type. In figure 11 the pressure ratio, for an ideal gas with $k = 1.405$ the propagation velocity of the compression shock and the change in entropy (expressed by π_{II}/π_I) has been presented as a function of the velocity change $\Delta v_{II,I}$. For compression shocks of the first type Δu and $\Delta v_{II,I}$ are to be taken with positive sign, for compression shocks of the second type with negative sign. The fundamental numerical values appear in table V. Such a diagram would have to be used to apply the characteristics method in the form given in section 11 in the computation of compression shocks. How are these relations for the compression shock expressed in terms of λ and μ? If two compression shocks which only arise separately from superposition of a velocity - they are distinguished by the indexes α and β - are represented in a $\lambda\mu$-diagram, that is, if the phases in advance of the shock $\lambda_{I,\alpha}$; $\mu_{I,\alpha}$; $\lambda_{I\beta}$; $\mu_{I\beta}$ and the phases behind the shock are plotted, then here, too, the expression must be arrived at that the thermodynamic phases in advance of and behind the shock, as well as the velocity differences for both compression shocks are the same. Accordingly,

$$\lambda_{I,\alpha} + \mu_{I,\alpha} = \lambda_{I\beta} + \mu_{I\beta}$$

$$\lambda_{II,\alpha} + \mu_{II,\alpha} = \lambda_{II\beta} + \mu_{II,\beta}$$

$$\left(\lambda_{II,\alpha} - \mu_{II,\alpha}\right) - \left(\lambda_{I,\alpha} - \mu_{I,\alpha}\right) = \left(\lambda_{II,\beta} - \mu_{II,\beta}\right) - \left(\lambda_{I\beta} - \mu_{I\beta}\right)$$

By subtraction of the first two equations

$$\left(\lambda_{II,\alpha} - \lambda_{I,\alpha}\right) + \left(\mu_{II,\alpha} - \mu_{I,\alpha}\right) = \left(\lambda_{II,\beta} - \lambda_{I\beta}\right) + \left(\mu_{II\beta} - \mu_{I\beta}\right)$$

Rearranging terms in the third equation gives

$$\left(\lambda_{II,\alpha} - \lambda_{I,\alpha}\right) - \left(\mu_{II,\alpha} - \mu_{I,\alpha}\right) = \left(\lambda_{II\beta} - \lambda_{I,\beta}\right) - \left(\mu_{II\beta} - \mu_{I\beta}\right)$$

From the last two equations it follows that

$$\lambda_{II,\alpha} - \lambda_{I,\alpha} = \lambda_{II,\beta} - \lambda_{I,\beta}$$

$$\lambda_{II,\alpha} - \mu_{I,\alpha} = \mu_{II,\beta} - \mu_{I,\beta}$$

that is, the changes in λ and μ in a compression shock are maintained in the superposition of a velocity. Accordingly, the shocks are designated by

$$\Delta\lambda_{II,I} = \lambda_{II} - \lambda_I$$

and

$$\Delta\mu_{II,I} = \mu_{II} - \mu_I$$

The following relations hold for ideal gases with constant specific heats, according to (42)

$$\Delta\lambda_{II,I} = \lambda_{II} - \lambda_I = \frac{2}{k-1}\left(\frac{a_{II}}{a_o} - \frac{a_I}{a_o}\right) + \frac{v_{II}}{a_o} - \frac{v_I}{a_o}$$

$$= \frac{a_I}{a_o}\left[\frac{2}{k-1}\left(\frac{a_{II}}{a_I} - 1\right) + \frac{v_{II} - v_I}{a_I}\right]$$

$$\Delta\mu_{II,I} = \mu_{II} - \mu_I = \frac{2}{k-1}\left(\frac{a_{II}}{a_o} - \frac{a_I}{a_o}\right) - \frac{v_{II}}{a_o} - \frac{v_I}{a_o}$$

$$= \frac{a_I}{a_o}\left[\frac{2}{k-1}\left(\frac{a_{II}}{a_I} - 1\right) - \frac{v_{II} - v_I}{a_I}\right]$$

a_I/a_o is to be computed from λ_I and μ_I by (42c). For the expressions in curved brackets

$$\Delta\bar{\lambda} = \left[\frac{2}{k-1}\left(\frac{a_{II}}{a_I} - 1\right) + \frac{v_{II} - v_I}{a_I}\right]$$

$$\Delta\bar{\mu} = \left[\frac{2}{k-1}\left(\frac{a_{II}}{a_I} - 1\right) - \frac{v_{II} - v_I}{a_I}\right]$$

are introduced. These quantities, as well as $\Delta u/a_I$, π_{II}/π_I, and p_{II}/p_I depend only on v_{Is}/a_I according to relations previously developed. They are plotted in figures 12(a) and 12(b), and, admittedly, the upper designations refer to the compression shocks of the first type, and the lower designations to compression shocks of the second type. Figure 12(b) represents an increased section of figure 12(a), with the appropriate numerical values in table V.

The following example shows a first application of this diagram. In a pipe of constant cross section there is a quiescent gas of constant entropy and constant pressure, the sonic velocity is taken to be $a_I = a_o$. Suddenly, a piston is driven into the pipe at a uniform speed of $0.5a_o$. What is the ensuing flow like? Figure 13 shows the yt-diagram. The starting point of the piston motion lies at the origin of the coordinate system. The life line of the piston is shown with hatching. A compression shock forms in front of the piston, which imparts the velocity of the piston to the particles, so that the particles behind the compression shock move with constant velocity. Corresponding to the phase in front of the compression shock is

$$\lambda_I = 0; \quad \mu_I = 0$$

The velocity behind the compression shock is

$$v_{II} = 0.5a_o$$

therefore,

$$\frac{1}{2}\left(\lambda_{II} - \mu_{II}\right) = 0.5$$

$$\lambda_{II} - \mu_{II} = 1$$

From this, on account of $\lambda_I = 0$ and $\mu_I = 0$

$$\Delta\lambda_{II,I} - \Delta\mu_{II,I} = 1$$

Since $a_I/a_o = 1$ this gives

$$\Delta\bar{\lambda} - \Delta\bar{\mu} = 1$$

As a result of causing this straight line in the $\Delta\bar{\lambda}\Delta\bar{\mu}$-diagram (fig. 12(b)) to intersect the shock curve, the following is obtained:

$$\Delta\bar{\lambda} = 1.022; \quad \Delta\bar{\mu} = 0.022; \quad \frac{\Delta u}{a_I} = 1.346; \quad \pi_{II}/\pi_I = 0.970$$

$$\lambda_{II} = 1.022; \quad \mu_{II} = 0.022; \quad u = 1.346$$

From λ_{II} and μ_{II}, P is computed by (42d), from this by (28)

$$p_{II}/p_I = 1.970$$

The goal would be reached somewhat quicker in this by application of diagram 11.

15. PRELIMINARY ARGUMENTS IN THE DETERMINATION OF A COMPRESSION SHOCK IN THE FLOW FIELD

It is the object of this section to show first of all by what data a compression shock in a flow is determined, and, secondly, to give a method by which the computation of such a compression shock is possible.

As can be readily shown, the velocity of a compression shock is larger than the velocity of a Mach wave in the material. This means, that the flow field in advance of the compression shock remains unaffected by this and can be computed independent of it. It will be assumed to be known what follows. For the field behind the shock, a compression shock of the first type represents on the one hand the start of life lines and Mach waves 2, on the other hand the terminal of Mach waves 1. It follows, from this, that the flow behind the shock and the shock itself are mutually related and can only be computed together. This is the reason, therefore, that the computation of the compression shocks becomes, essentially, more complicated than the computation of other parts of the flow.

Next will be shown how examples can be conceived of flow fields with compression shocks. If in the yt-diagram (figs. 14(a) and 14(b) the flow field in front of the compression shocks and the portion CD of the life line of the compression shock is given, then the phases behind the shock are also determined. From the slope of the life line the propagation velocity of the compression shock is given, namely for each point of CD. Beside, the phases in front of the shock can be learned for the points of CD; with this the phases behind the shock are calculable. From the phases behind the shock, a portion of the flow field behind the shock, namely the region CED (fig. 14(a)) may be computed, or if the entropy is known for the life lines at the lower end of C. The region CFD (fig. 14(b)) as well. It is necessary to go forward along the life lines and Mach waves 2, backwards along Mach waves 1. Imagine in figure 14(a) that the computed life line CE is realized through the motion of a piston, then there is a flow in which a compression shock appears and which satisfies a boundary condition (if not prescribed, too). In figure 14(b) it is necessary to imagine another flow field adjoined continuously at the lower end of CF; here the compression shock and the flow determined by it satisfy the condition that it is compatible along the Mach wave CF with another flow.

From these flow fields the following is recognized; the compression shock through the portion CE of the life line of the piston or CF of the Mach wave is defined as far as it is reached by Mach waves of its type (here the first, therefore). A change of the life line of the piston outside of CE or the Mach waves outside of CF propagates along Mach wave 1 in the yt-diagram, to be exact, and neglecting cases in which a second compression shock arises, attains the compression shock at the upper end of D, certainly. On the other hand a change brought about between C and E or between C and F in the boundary or junction conditions takes effect at that position on the compression shock where the Mach wave 1 concerned reaches it, that is, the portion CD is certainly changed.

If the life line of the piston is known beyond E to G or the Mach wave beyond F to H, then a further portion of the flow field is thereby determined, without the necessity for knowing the continuation of the compression shock beyond D; it concerns the regions CEGJD or CFHKD.

It will now be shown how to procede fundamentally to compute a compression shock for specified boundary or junction conditions. As a concrete example assume the compression shock to be produced by a piston which experiences a sudden jump in velocity. (See fig. 15.) The starting point of the compression shock is that point of the life line of the piston at which the velocity jump appears. The phase immediately behind M can be ascertained immediately by the method applied to the example of the last section. The compression shock - as in previous examples of Mach waves - is computed in individual sections, which are so small that the phase quantities for them may be regarded as varying linearly. As just carried out, the phases behind the compression shock are calculable, if the velocity of the shock is known. The velocity at M is known. Along the portion of the compression shock to be computed, M, N, the phase change and, with it, the change in propagation velocity of the compression shock, too, are regarded as linear. Accordingly, for all possible shocks which satisfy the transition conditions, the portion M, N, of the compression shock depends only on a single parameter, the velocity change between M and N, to be exact. As a result of computing the field behind the compression shock for various values of this parameter, by interpolation, that shock may be ascertained which is consistent with the specified piston movement. At best, for this N is permitted to travel on a fixed life line in the field in advance of the shock. Let C be the point on the life line for which the Mach wave 1 passing through N proceeds. Now the region OPQN may be computed in a familiar manner. For the determination of the extension of the compression shock NR the phase behind the compression shock at the point N may be regarded as given everywhere along the entire Mach wave NQ. On the other hand, that value of velocity changes between N and R has to be determined by interpolation, which relates to a flow field that continuously joins the known field along NQ.

With these two types, namely the computation of a compression shock going out from a piston or wall and the computation of a compression shock continuing into or arising in the interior of the flow, the most important problems have been mastered that can appear here. The interpolation methods described become pretty tedious; instead of them, iteration methods will be used, which actually lead to the goal more quickly. The interpolation method was mentioned previously, however, since it affords better insight into the basic relations.

16. EXAMPLES OF THE COMPUTATION OF COMPRESSION SHOCKS IN THE FLOW FIELD

Examples will be given of how the problems formulated in the preceding section can be solved by means of iteration methods. Let the flow be that computed in figure 10 and table IV. As the start of the new portion of the compression shock to be computed, point 1 is chosen in every case, accordingly it is identified with the point M (fig. 15) once and with the point N a second time. The new portion of the compression shock to be computed that corresponds to MN or NR, accordingly, is assumed to end on the life line 8, 9 of figure 10. The phases in front of the shock for N or R are obtained as a result of interpolation along this line. For these calculations it is necessary, on that account, to have the knowledge of the flow field in front of the shock at the points 1 (M or N) and 8 and 9. In table VI which has the same arrangement as table IV these values have been recorded. While it sufficed to know $\frac{d\pi}{d\psi}$ for the construction of the flow field, here π itself must be known. These quantities for points 1, 8, and 9 are located in column 26. In the designations, in these examples, the only deviation from figure 15 is that only points on the compression shock are characterized by letters. Numbers are used for points of the flow field, corresponding to previous use.

We begin with the more elementary problem of continuing a compression shock in the interior of the flow. For this the phase behind the shock at the point N and the phases along the Mach wave N_{II},10 (fig. 16(a)) may be considered known. The phases at N_{II} and at point 10 appear in table VI, phases in between are found by linear interpolation; moreover, for N_{II} the velocity of the compression shock and π have been given. (columns 25 and 26). Besides $\frac{d\pi}{d\psi}$ for the life lines lying below N may be viewed as computed. It was entered for point 10 in the corresponding column. If the distances between points on the compression shock are not chosen too large, it is sufficient to regard $\frac{d\pi}{d\psi}$ between them as as constant. In the following this has happened throughout. Since N_{II} and 10 lie on a Mach wave, the consistency condition must naturally be satisfied.

In connection with the flow calculation the existing data are to be taken from the preceding calculation steps. The real computation begins with the fact that the difference in ψ from its value at the starting point of the portion of the compression shock to be computed (N here) is ascertained for the life line up to which the compression shock is to be computed (8, 9 here). This computation is carried through along the curve of the initial values in figure 10, the life line 8, 9 used here passes through point 7 there. By (37)

$$\Delta\psi_{7,N} = \int_N^7 \left(\frac{F}{F_o}\frac{\rho}{\rho_o}dy - \frac{F}{F_o}\frac{\rho}{\rho_o}v\,dt\right) = \left(\frac{F}{F_o}\frac{\rho}{\rho_o}\right)_{mN,7} \Delta y_{7,N}$$

$$-\left(\frac{F}{F_o}\frac{\rho}{\rho_o}\frac{v}{a_o}\right)_{mN,7} \Delta a_o t_{7,N}$$

By (23) and (42d)

$$\frac{\rho}{\rho_o} = \pi\left[1 + \frac{k-1}{4}(\lambda + \mu)\right]^{\frac{2}{k-1}}$$

Just as for figure 10, F has the form

$$F = F_o y^2 t$$

For point 7

$y = 1.450;\quad a_o t = 1.180;\quad \lambda = 0.66;\quad \mu = -0.16;\quad \pi_7 = \pi_8 = 0.849$

For N the corresponding values appear in table VI. With this the following is obtained:

$$\left(\frac{F}{F_o}\frac{\rho}{\rho_o}\right)_7 = 2.680;\qquad \left(\frac{F}{F_o}\frac{\rho}{\rho_o}\frac{v}{a_o}\right)_7 = 1.110$$

$$\left(\frac{F}{F_o}\frac{\rho}{\rho_o}\right)_N = 2.170;\qquad \left(\frac{F}{F_o}\frac{\rho}{\rho_o}\frac{v}{a_o}\right)_N = 0.930$$

$$\left(\frac{F}{F_o}\frac{\rho}{\rho_o}\right)_{mN,7} = 2.425;\qquad \left(\frac{F}{F_o}\frac{\rho}{\rho_o}\frac{v}{a_o}\right)_{mN,7} = 1.020$$

$$\Delta\psi_{7,N} = 2.425 \times 0.075 + 1.020 \times 0.03 = 0.2122$$

In figure 10 $\frac{d\pi}{d\psi}$ had already been given, it must be the same as that found from the quantities just computed. In fact

$$\left(\frac{d\pi}{d\psi}\right)_{m7,N} = \frac{\pi_7 - \pi_N}{\Delta\psi_{7,N}} = \frac{0.049}{0.212} = 0.230$$

This is the average value of $\frac{d\pi}{d\psi}$ as can be gathered for the stretch 1.7 from the auxiliary diagram in figure 10. After these preparations, the actual iteration method is reached. To begin with, the phases at the points R_{II} and 11 are estimated, in that 11 is the intersection point of the Mach wave 1 leading backwards from R with the given Mach wave N,10. Since no better reference point exists for the estimate, these phases are equated to the phase at N_{II}. Moreover, still another estimate is needed for $\frac{d\pi}{d\psi}$ behind the shock; for this, the same value that prevails at the lower end of N is chosen. With these assumptions, the figure N, R, 11 may be drawn in figure 16(a). Starting with the life line of the compression shock NR, whose direction here is the same as the direction of the compression shock at N (table VI), R is obtained as the intersection point with the life line 8, 9. Then the Mach wave R,11 is drawn in proceeding from R backwards. The direction of this Mach wave was taken in the familiar manner from a λμ-diagram (not given here). From this figure the position of R in advance of the shock is learned by interpolation along N,10 the phase at 11. (See table VI.) From this may be obtained the values entered further on in the respective lines which are necessary for later computation. Proceeding from λ_{II} by means of the consistency conditions, the quantity λ_{RII} is computed for the Mach wave $(11,R_{II})$. For this the initial estimates for the phase in R_{II} are taken as a basis and then columns 6 to 13, 17, 15, 16 and 18 to 20 computed. For λ_{RII} so obtained the properties of the compression shock are taken from the shock diagram 12(b). The following computations are essential to this

$$\Delta\lambda_{RII,I} = \lambda_{RII} - \lambda_{RI} = 0.954$$

$$\Delta\bar{\lambda}_R = \Delta\lambda_{RII,I}\Big/\left(a_{RI}/a_o\right) = 0.954/1.040 = 0.918$$

From the shock diagram

$$\Delta\bar{\mu}_R = 0.0130; \quad \pi_{RII}/\pi_{RI} = 0.978; \quad \frac{\Delta u_R}{a_{RI}} = 1.307$$

From this it is computed that

$$\Delta\mu_{RII,I} = 0.0135; \quad \pi_{RII} = 0.830; \quad \frac{\Delta u_R}{a_o} = 1.360$$

$$\mu_{RII} = -0.090; \quad u_R/a_o = v_{RI}/a_o + \Delta u_R/a_o = 1.661$$

A portion of these results are given in table VI (columns 24 to 26). Moreover

$$\frac{d\pi}{d\psi} = \frac{\pi_{RII} - \pi_{NII}}{\Delta\psi_{R,N}} = \frac{0.830 - 0.781}{0.2122} = 0.230$$

To improve these values, let a second iteration step be carried out. First, the figure N,R,11 has to be drawn again for the values just obtained. The average direction of the compression shock is

$$u_{mN,R} = \frac{1}{2}(u_N + u_R) = 1.733$$

Then R_I and 11 are obtained by interpolation, λ_{RII} from the consistency condition for the Mach wave $11,R_{II}$.

To find the characteristics of the shock, it is necessary to carry out the following computation

$$\Delta\lambda_{RII,I} = 1.456 - 0.493 = 0.963; \quad \Delta\bar{\lambda}_R = 0.927$$

From the shock diagram

$$\Delta\bar{\mu} = 0.0130; \quad \pi_{RII}/\pi_{RI} = 0.980; \quad \frac{\Delta u_R}{a_{RI}} = 1.310$$

From this is obtained

$$\mu_{R,II} = -0.087; \quad \pi_{RII} = 0.828; \quad u_{RII}/a_o = 1.657; \quad \frac{d\pi}{d\psi} = 0.220$$

An additional iteration step is not necessary any more. In the second example (fig. 16(b)) the compression shock is produced by the sudden velocity change of a piston. The point of the yt-diagram at which this velocity jump takes place - let it be designated M in agreement with figure 15 - is to coincide with point 1 of figure 10. From the point M the piston has the velocity corresponding to the life line in the field in front of the shock, in particular the velocity at M in front of the velocity jump is $0.425a_o$. At M the velocity changes, suddenly, to the value $v_{M_{II}} = 0.925a_o$ and rises until the instant $a_o t = 1.3$ to the magnitude $0.975a_o$. This and the flow field as determined by the initial conditions and the piston motion up to the point M is given. Next the phase behind the shock at the point M is computed.

$$\frac{\Delta v_{MII,I}}{a_o} = \left(\frac{v}{a_o}\right)_{M,I} - \left(\frac{v}{a_o}\right)_{M,II} = 0.500$$

$$\frac{\Delta v_{MII,I}}{a_{M_I}} = 0.483$$

$$\frac{1}{2}(\Delta\bar{\lambda} - \Delta\bar{\mu}) = 0.483$$

As a result of this line in the shock diagram 12(b) intersecting the shock curve, the following is obtained

$$\Delta\bar{\lambda} = 0.986; \quad \Delta\bar{\mu} = 0.020$$

$$\pi_{M,II}/\pi_{MI} = 0.977; \quad \frac{\Delta u_M}{a_{MI}} = 1.333$$

From this

$$\lambda_{M,II} = 1.620; \quad \mu_{M,II} = -0.229$$

$$\pi_{M,II} = 0.781; \qquad \frac{u_M}{a_o} = 1.805$$

The phase at M_{II} is known with that. (table VI.) Now the difference must be computed, over again, from the life line of the piston for the life line up to which it is desired to compute the compression shock. It is desired to allow the compression shock to end at the life line 8, 9, here too and take the phases in 8 and 9 (table VI) from the preceding example and

$$\Delta\psi_{8,M} = \Delta\psi_{N,M} = 0.2122$$

The computation of the compression shock makes use of figure M,S, N, 11. (See fig. 16(b)). M, S, N is the life line of the compression shock; N, 11 is the Mach wave 1 returning from N; 11, S is the Mach wave 2 returning from 11. To begin, an estimate of the phase at the points N_{II}, 11 and S_{II} is made and this is chosen equal everywhere to the phase at M_{II}. In addition, an estimate for $\frac{d\pi}{d\psi}$ is necessary. Let $\frac{d\pi}{d\psi} = 0.230$ as a start. Figure M, N, 11, S may be drawn with these assumed values. The order in which the points were named corresponds to the order in which they came up in the drawing. For the positions of N and 11 obtained thereby the phase in front of the shock (see table VI) or the velocity of the life line is obtained by interpolation. The iteration method begins at point 11 and it can be shown that μ_{11} can be only slightly different from $\mu_{S_{II}}$ because the line element S_{II},11 is small relative to the other dimensions. The quantity $\mu_{S_{II}}$ can differ from $\mu_{M_{II}}$ only slightly, since it originates in linear interpolation between M and N, and N lies very close to M. Therefore $\mu_{11} = \mu_{M,II}$ is chosen as a starting point. If the velocity of the piston at 11 that is known from the boundary conditions is used for this λ_{11} may be computed. From the consistency condition for the Mach wave 11,N $\lambda_{N_{II}}$ is obtained. Now the following computation

$$\Delta\lambda_{NII,I} = 1.017 \qquad \Delta\bar{\lambda}_N = 0.988$$

and from the shock diagram

$$\Delta\bar{\mu}_N = 0.020; \quad \pi_{NII}/\pi_{NI} = 0.973; \quad \frac{\Delta u_N}{a_{NI}} = 1.334$$

from this

$$\mu_{N,II} = -0.077; \quad \pi_{NII} = 0.827; \quad \frac{u_N}{a_o} = 1.678$$

Further it is calculated that

$$\frac{d\pi}{d\psi} = 0.217$$

The phase at S is obtained by interpolation between M and N. With the aid of the consistency condition for the Mach wave $S,11,\mu_{11}$ is finally obtained, and λ_{11} from the boundary condition for point 11. The first iteration step ends with that. It is necessary to check whether the quantities λ_{11}, μ_{11}, λ_{NII}, μ_{NII}, u_{NII}, and $\frac{d\pi}{d\psi}$ computed agree sufficiently with the original estimates.

To increase the accuracy a second iteration step might be carried out. On the basis of the values just computed, the figure is redesigned and the computation is carried out in the manner just described. The value for μ_{11} just computed is taken as a beginning. The following calculation is obtained for the determination of the characteristics of the shock

$$\Delta\lambda_{NII,I} = 1.003; \quad \Delta\bar{\lambda}_N = 0.967$$

From the shock diagram

$$\Delta\bar{u}_N = 0.018; \quad \pi_{NII}/\pi_{N,I} = 0.975; \quad \frac{\Delta u_N}{a_{NI}} = 1.327$$

from this

$$\mu_{NII} = -0.083; \quad \pi_{NII} = 0.828; \quad \frac{u_N}{a_o} = 1.678; \quad \frac{d\pi}{d\psi} = 0.221$$

The computation is continued in the manner given until the phase at point 11 is obtained, again. An additional iteration step is not necessary.

17. SUMMARY

The differential equation system for nonstationary, one-dimensional flows possesses three families of characteristics; the thermodynamic and the flow phase are described by three variables. As a result of setting up consistency conditions for the characteristics passing through the point for which the conditions have been set up, three equations are obtained from which the phase may be obtained. In that a possibility for the computation of the flow has been given fundamentally. The report carries out these ideas, in general, and brings the simplifications which are possible under special assumptions, as well as detailed examples. Compression shocks appear, in this, as transitional conditions in the interior of the flow and are likewise investigated in detail.

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TABLE I

DIRECTION OF THE MACH WAVES IN ISENTROPIC FLOWS IN

THE PIPE OF CONSTANT CROSS SECTION

All values ±

	$\frac{\lambda}{\mu} = 0.1$	$\frac{\lambda}{\mu} = 0.2$	$\frac{\lambda}{\mu} = 0.3$	$\frac{\lambda}{\mu} = 0.4$	$\frac{\lambda}{\mu} = 0.5$	$\frac{\lambda}{\mu} = 0.6$	$\frac{\lambda}{\mu} = 0.7$	$\frac{\lambda}{\mu} = 0.8$	$\frac{\lambda}{\mu} = 0.9$	$\frac{\lambda}{\mu} = 1.0$	$\frac{\lambda}{\mu} = 1.1$	$\frac{\lambda}{\mu} = 1.2$	$\frac{\lambda}{\mu} = 1.3$	$\frac{\lambda}{\mu} = 1.4$
$\frac{\mu_m}{\lambda_m} = 0.05$	1.034	1.090	1.146	1.203	1.259	1.315	1.372	1.428	1.484	1.540	1.596	1.653	1.709	1.765
$\frac{\mu_m}{\lambda_m} = 0.15$	.990	1.046	1.103	1.159	1.215	1.272	1.328	1.384	1.440	1.496	1.553	1.609	1.665	1.722
$\frac{\mu_m}{\lambda_m} = 0.25$	.946	1.003	1.059	1.115	1.172	1.228	1.284	1.340	1.396	1.453	1.509	1.565	1.622	1.678
$\frac{\mu_m}{\lambda_m} = 0.35$	.903	.959	1.015	1.072	1.128	1.184	1.240	1.296	1.353	1.409	1.465	1.522	1.578	1.634
$\frac{\mu_m}{\lambda_m} = 0.45$	.859	.915	.972	1.028	1.084	1.140	1.196	1.253	1.309	1.365	1.422	1.478	1.534	1.590
$\frac{\mu_m}{\lambda_m} = 0.55$	.815	.872	.928	.984	1.040	1.096	1.153	1.209	1.265	1.322	1.378	1.434	1.490	1.547
$\frac{\mu_m}{\lambda_m} = 0.65$	.772	.828	.884	.940	.996	1.053	1.109	1.165	1.222	1.278	1.334	1.390	1.447	1.502
$\frac{\mu_m}{\lambda_m} = 0.75$	.728	.784	.840	.896	.953	1.009	1.065	1.122	1.178	1.234	1.290	1.347	1.402	1.459
$\frac{\mu_m}{\lambda_m} = 0.85$	.684	.740	.796	.853	.909	.965	1.022	1.078	1.134	1.190	1.247	1.302	1.359	1.415
$\frac{\mu_m}{\lambda_m} = 0.95$	.640	.696	.753	.809	.865	.922	.978	1.034	1.090	1.147	1.202	1.259	1.315	1.370
$\frac{\mu_m}{\lambda_m} = 1.05$	.596	.653	.709	.765	.822	.878	.934	.990	1.047	1.102	1.159	1.215	1.270	1.327
$\frac{\mu_m}{\lambda_m} = 1.15$	.553	.609	.665	.722	.778	.834	.890	.947	1.002	1.059	1.115	1.170	1.227	1.283
$\frac{\mu_m}{\lambda_m} = 1.25$	.509	.565	.622	.678	.734	.790	.847	.902	.959	1.015	1.070	1.127	1.183	1.238
$\frac{\mu_m}{\lambda_m} = 1.35$	.465	.522	.578	.634	.690	.747	.802	.859	.915	.970	1.027	1.083	1.138	1.194
$\frac{\mu_m}{\lambda_m} = 1.45$	.422	.478	.534	.590	.647	.702	.759	.815	.870	.927	.983	1.038	1.094	1.200

TABLE II

ISENTROPIC FLOW OF ANY GAS IN THE PIPE OF VARIABLE CROSS SECTION

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
Point	Originates from		λ estimated	μ estimated	$\lambda + \mu$	$\frac{a}{a_o}$	$\frac{v}{a_o}$	$\frac{v+a}{a_o}$	$\frac{v-a}{a_o}$	M	y	$a_o t$	$(v+a)_m$	M_m	$\Delta(a_o t)$	$\Delta\lambda$	λ calculated	$(v-a)_m$	M_m	$\Delta(a_o t)$	$\Delta\mu$	μ calculated
1					0.350	1.021	0.425	1.446	-0.596	1.475	1.375	1.210					0.600					-0.250
2					.600	1.037	.400	1.437	-.637	1.440	1.500	1.166					.700					-.100
3					.850	1.053	.375	1.428	-.678	1.412	1.625	1.137					.800					.050
4	1	2	0.600	-0.100	.500	1.030	.350	1.380	-.680	1.322	1.440	1.254	1.413	1.398	0.044	-0.062	.538	-0.659	1.381	0.088	-0.122	-.222
			.538	-.222	.316	1.020	.380	1.400	-.640	1.348	1.444	1.256	1.423	1.410	.046	-.065	.535	-.638	1.395	.090	-.125	-.225
5	2	3	.630	-.030	.600	1.037	.330	1.367	-.707	1.288	1.568	1.217	1.402	1.364	.051	-.069	.630	-.693	1.350	.079	-.107	-.057
			.630	-.057	.577	1.036	.344	1.380	-.692	1.305	1.570	1.217	1.409	1.382	.051	-.071	.629	-.685	1.359	.080	-.108	-.059
6	4	5	.480	-.160	.320	1.019	.320	1.339	-.699	1.213	1.507	1.306	1.360	1.280	.050	-.064	.471	-.696	1.259	.088	-.111	-.170
			.477	-.170	.391	1.018	.318	1.338	-.698	1.210	1.507	1.306	1.359	1.280	.050	-.064	.471	-.696	1.258	.088	-.111	-.170

TABLE III

NONISENTROPIC FLOW OF AN IDEAL GAS WITH VARYING SPECIFIC HEAT

Treatment of Boundary Conditions

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Point	Originates from		λ estimated	μ estimated	$\lambda + \mu$	a/a_o	v/a_o	$\frac{v+a}{a_o}$	$\frac{v-a}{a_o}$	y	$a_o t$	$\frac{dx}{dt}$	P	M	N	-M-N	-M+N	$\left(\frac{v+a}{a_o}\right)_m$	$(-M-N)_m$	$\Delta a_o t$	$\Delta\lambda$	λ calculated	$\left(\frac{v-a}{a_o}\right)_m$	$(-M+N)_m$	$\Delta a_o t$	$\Delta\mu$	μ calculated
1					0.350	1.021	0.425	1.446	-0.596	1.375	1.210	0.240	1.200	1.474	0.505	-1.979	-0.969					0.600					-0.250
2					.600	1.037	.400	1.437	-.637	1.500	1.166	.201	1.400	1.442	.536	-1.978	-.906					.700					-.100
3					.850	1.053	.375	1.428	-.678	1.625	1.138	.165	1.680	1.412	.640	-2.052	-.772					.800					.050
4	1	2	0.600 .504	-0.100 -.176	.500 .328	1.031 1.020	.350 .340	1.381 1.361	-.681 -.679	1.440 1.438	1.259 1.260	.230 .230	1.300 1.200	1.320 1.290	.568 .552	-1.918 -1.842	-.722 -.758	1.414 1.404	-1.949 -1.911	0.049 .050	-0.096 -.096	.504 .504	-0.659 -.658	-0.814 -.832	0.093 .094	-0.076 -.078	-.176 -.178
5	2	3	.600	-.010	.590	1.037	.305	1.342	-.732	1.567	1.220	.189	1.400	1.253	.608	-1.861	-.645	1.390	-1.920	.054	-.104	.596	-.705	-.709	.082	-.059	-.008
6	4	5	.405 .420	-.073 -.059	.332 .361	1.021 1.022	.239 .240	1.259 1.262	-.782 -.782	1.499 1.499	1.308 1.308	.215 .215	1.210 1.210	1.107 1.110	.587 .587	-1.694 -1.697	-.520 -.523	1.311 1.312	-1.768 -1.770	.048 .048	-.085 -.085	-.420 -.420	-.757 -.757	-.583 -.584	.088 .088	-.051 -.051	-.059 -.059
7	5		.800 .417 .423	.050 -.229 -.219	.850 .188 .204	1.053 1.012 1.012	.375 .323 .323 .321 .321 .321	1.428 1.335 1.333	 -.691	1.686 1.688 1.688	1.313 1.318 1.318	.165 .165 .165	1.680 1.100 1.100	1.206 1.150 1.150	.794 .523 .523	-2.000 -1.673 -1.673	-.412 -.628 -.628	1.385 1.338 1.338	-1.926 -1.767 -1.767	.093 .098 .098	-.179 -.173 -.173	.417 .423 .423					-.229 -.219 -.219
8	6	7	.250 .248	-.210 -.270	.040 -.022	1.002 1.001	.230 .259	1.232 1.260	-.772 -.742	1.618 1.618	1.414 1.413	.189 .189	1.020 1.010	.993 1.029	.548 .542	-1.541 -1.571	-.445 -.487	1.247 1.260	-1.619 -1.634	.106 .105	-.172 -.172	.248 .248	-.732 -.717	-.536 -.577	.096 .095	-.051 -.053	-.270 -.272
9	8		.120 .086	-.440 -.434	-.320 -.348	.979 .978	.280 .260 .260 .259	1.259 1.238		1.747 1.748	1.520 1.526	.165 .165	.900 .900	.936 .857	.528 .531	-1.464 -1.388	-.403 -.326	1.296 1.294	-1.518 -1.480	.107 .113	-.162 -.167	.086 .081					-.434 -.437
10		4	.600 .625	-.250 -.291	.350 .334	1.021 1.021	.425 .458	 1.479	-.596 -.563	1.375 1.375	1.359 1.363		1.200 1.210	1.382 1.427		-1.382 -1.427	-1.388 -1.427					.625 .630	-.638 -.621	-1.142 -1.167	.099 .103	-.113 -.121	-.291 -.299
12	4	6	.555 .575	-.195 -.121	.360 .454	1.021 1.028	.375 .348	1.396 1.376	-.646 -.680	1.432 1.431	1.404 1.403		1.200 1.280	1.260 1.233		-1.260 -1.233	-1.260 -1.233	1.438 1.428	-1.344 -1.330	.041 .040	-.055 -.053	.575 .577	-.714 -.731	-.647 -.643	.096 .095	-.062 -.060	-.121 -.118
13		12	.590 .596	-.210 -.226	.380 .370	1.023 1.022	.400 .411		-.623 -.611	1.375 1.375	1.489 1.489		1.220 1.210	1.283 1.297		-1.283 -1.297	-1.283 -1.297					.596 .595	-.652 -.646	-1.258 -1.265	.086 .086	-.108 -.109	-.226 -.227

TABLE IV

NONISENTROPIC FLOW OF AN IDEAL GAS WITH CONSTANT SPECIFIC HEAT

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
Point	Originates from		λ estimated	μ estimated	λ + μ	$\frac{V}{a_o}$	$\frac{a}{a_o}$	$\left(\frac{a}{a_o}\right)^7$	$\frac{dx}{d\psi}$	M	N	−M − N	−M + N	y	$a_o t$	$(-M - N)_m$	$\Delta a_o t$	Δλ	λ calculated	$(-M + N)_m$	$\Delta a_o t$	Δμ	μ calculated
1					0.350	0.425	1.035	1.272	0.240	1.496	0.498	−1.994	−0.998	1.375	1.210				0.600				−0.250
2					.600	.400	1.060	1.503	.201	1.473	.567	−2.040	−.907	1.500	1.166				.700				−.100
3					.850	.375	1.085	1.770	.165	1.454	.627	−2.081	−.827	1.625	1.138				.800				.050
4	1	2	0.600	−0.100	.500	.350	1.050	1.407	.227	1.348	.593	−1.941	−.755	1.439	1.254	−1.968	0.044	−0.087	.514	−0.831	0.088	−0.073	−.173
			.514	−.173	.341	.344	1.034	1.263	.230	1.320	.538	−1.858	−.782	1.438	1.255	−1.926	.045	−.087	.513	−.844	.089	−.075	−.175
5	2	3	.615	−.020	.595	.318	1.059	1.499	.190	1.300	.609	−1.909	−.691	1.570	1.213	−1.974	.047	−.093	.607	−.759	.075	−.057	−.007
			.607	−.007	.600	.307	1.060	1.503	.190	1.287	.610	−1.897	−.677	1.570	1.213	−1.968	.047	−.093	.606	−.752	.075	−.056	−.006
6	4	5	.410	−.075	.335	.243	1.034	1.259	.213	1.130	.562	−1.692	−.568	1.502	1.300	−1.775	.045	−.079	.433	−.623	.087	−.054	−.060
			.433	−.060	.373	.247	1.037	1.292	.213	1.136	.567	−1.713	−.559	1.502	1.300	−1.786	.045	−.080	.433	−.618	.087	−.054	−.060

TABLE V

NONSTATIONARY COMPRESSION SHOCKS (κ = 1.405)

$a_I/\Delta u$	$\Delta u/a_I$	$\Delta v_{II,I}/a_I$	p_{II}/p_I	$\varkappa_{II}/\varkappa_I$	$\frac{\Delta\bar{\lambda}}{\Delta\bar{\mu}}$	$\frac{\Delta\bar{\mu}}{\Delta\bar{\lambda}}$
1.00	1.00	0.0	1.00	1.00	0	0
0.98	1.02041	0.03360	1.04817	0.99998	0.06721	0.00001
.96	1.04167	.06791	1.09938	.99991	.13588	.00006
.94	1.06383	.10298	1.15391	.99970	.20615	.00019
.92	1.08696	.13884	1.21203	.99927	.27816	.00047
.90	1.11111	.17556	1.27406	.99862	.35206	.00094
.88	1.13636	.21319	1.34037	.99748	.42806	.00167
.86	1.16279	.25180	1.41137	.99587	.50633	.00273
.84	1.19048	.29146	1.48748	.99369	.58711	.00419
.82	1.21951	.33224	1.56925	.99080	.67062	.00615
.80	1.25	.37422	1.65722	.98708	.75713	.00869
.78	1.28205	.41751	1.75204	.98240	.84694	.01193
.76	1.31579	.46220	1.85445	.97659	.94038	.01599
.74	1.35135	.50840	1.96527	.96951	1.03781	.02101
.72	1.38889	.55625	2.08544	.96101	1.13965	.02716
.70	1.42857	.60588	2.21608	.95091	1.24634	.03458
.68	1.47059	.65745	2.35841	.93899	1.35843	.04345
.66	1.51515	.71114	2.51387	.92506	1.47647	.05418
.64	1.5625	.76715	2.68413	.90903	1.60114	.06684
.62	1.61290	.82570	2.87113	.89067	1.73320	.08180
.60	1.66667	.88704	3.07714	.86974	1.87350	.09941
.58	1.72414	.95147	3.30484	.84602	2.02302	.12009
.56	1.78571	1.01931	3.55735	.81954	2.18291	.14430
.54	1.85185	1.09094	3.83845	.79006	2.35447	.17260
.52	1.92308	1.16680	4.15261	.75746	2.53923	.20563
.50	2.0	1.24740	4.50520	.72182	2.73895	.24414
.48	2.08333	1.33333	4.90276	.68305	2.95572	.28905
.46	2.17391	1.42529	5.35333	.64131	3.19120	.34141
.44	2.27273	1.52410	5.86637	.59678	3.45070	.40251
.42	2.38095	1.63073	6.45517	.54980	3.73531	.47385
.40	2.5	1.74636	7.13409	.50078	4.05005	.55732
.38	2.63158	1.87242	7.92319	.45026	4.40174	.65517
.36	2.77778	2.01063	8.84701	.39889	4.79144	.77018
.34	2.94118	2.16314	9.93888	.34751	5.23210	.90582
.32	3.125	2.33264	11.2418	.29698	5.73172	1.06644
.30	3.33333	2.52252	12.8138	.24825	6.30261	1.25756
.28	3.57143	2.73716	14.7347	.20234	6.96061	1.48630
.26	3.84615	2.98225	17.1156	.16016	7.72656	1.76206
.24	4.16667	3.26542	20.1167	.12251	8.62809	2.09725
.22	4.54545	3.59705	23.9721	.090026	9.70301	2.50891
.20	5.0	3.99169	29.0416	.063089	11.0043	3.02088
.18	5.55556	4.47032	35.8933	.041760	12.6082	3.66757
.16	6.25	5.06445	45.4722	.025786	14.6295	4.50062
.14	7.14286	5.82359	61.6909	.014596	17.2482	5.60104
.12	8.33333	6.83019	80.9708	.007388	20.7642	7.10387
.10	10.0	8.23285	116.672	.003219	25.7176	9.25191
.08	12.5	10.3285	182.394	.001133	33.1878	12.53078
.06	16.6667	13.8101	324.386	.000288	45.6935	18.07325
.04	25	20.7568	730.082	.000040	70.7902	29.27672
.02	50	41.5634	2920.83	.000001	146.254	63.12755
.00	∞	∞	∞	0	∞	∞

TABLE VI

DETERMINATION OF COMPRESSION SHOCKS

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18.	19	20	21	22	23	24	25	26	
Point	Originates from		Estimated λ	μ	$\lambda + \mu$	v/a_o	a/a_o	$(a/a_o)^7$	$\frac{dx}{dy}$	M	N	-M - N	-M + N	y	$a_o t$	$(-M+N)_m$	$\Delta(a_o t)$	$\Delta\lambda$	λ	$(-M-N)_m$	$\Delta a_o t$	$\Delta\mu$	μ	x	u/a_o	
N_I														1.375	1.210				0.6				-0.25	0.8		
8															1.210				.591				-.144	.849		
9															1.295				.448				-.081	.849		
N_{II}														1.375	1.210				1.620				-.229	.781	1.805	
10									0.200					1.368	1.300				1.750				-.379			
R_I					0.396	0.301	1.040							1.478	1.268	Interpolation between 8,9			.499	Interpolation between 8,9			-.103	.849		First
11					1.413	.925	1.141	2.50	.200	2.595	0.708	-3.303		1.274	1.218	Interpolation between N_{II}, 10			1.631	Interpolation between N_{II}, 10			-.218			
R_{II}	11		1.620	-0.229	1.391	.925	1.139	2.48	.200	2.314	.980	-3.396		1.478	1.268	-3.350	0.050	-0.178	1.453	Shock diagram			-.090	.830	1.661	
R_I					.393	.297	1.039							1.479	1.271	Interpolation between 8,9				Interpolation between 8,9			-.100	.849		Second Step of Iteration
11					1.413	.925	1.141	2.50	.230	2.595	.813	-3.408		1.274	1.218	Interpolation between N_{II}, 10				Interpolation between N_{II}, 10			-.218			
R_{II}	11		1.453	-.090	1.363	.772	1.136	2.42	.230	2.080	1.111	-3.190		1.479	1.271	-3.300	0.053	-0.175	1.456	Shock diagram			-.087	.828	1.657	
N_I					.350	.425	1.035							1.375	1.210				.6				-.25	.8		
8															1.210				.591				-.144	.849		
9															1.295				.448				-.081	.849		
N_{II}						.985								1.375	1.210				1.620				-.229	.781	1.805	
10						.975									1.300											
N_I					.388	.292	1.039							1.478	1.268	Interpolation between 8,9			.486	Interpolation between 8,9			-.098			First
11					1.410	.934	1.141	2.50	.220	2.157	.927	-3.08		1.388	1.226	From boundary condition			1.639	Estimated			-.229			
N_{II}	11		1.620	-.229	1.391	.985	1.139	2.48	.220	2.314	1.080	-3.39		1.478	1.268	-3.24	0.042	-0.1362	1.503	Shock diagram			-.077	.827	1.678	
S_{II}					1.396	.905	1.140	2.49	.217	2.420	.900		-1.520	1.390	1.219	Interpolation between M_{II} and N_{II}			1.603	Interpolation between M_{II} and N_{II}			-.207			
11		S_{II}	1.639	-.229	1.410	.934	1.141	2.50	.217	2.467	.916		-1.551	1.388	1.226	From boundary condition			1.650	-1.535	0.007	-0.011	-.218			
N_I					.393	.288	1.039							1.478	1.270	Interpolation between 8, 9			.495	Interpolation between 8,9			-.102			Second Step of Iteration
11					1.430	.933	1.143	2.56	.217	2.470	.950	-3.12		1.388	1.225	From boundary condition			1.645	Result of former step of iteration			-.218			
N_{II}	11		1.503	-.077	1.426	.790	1.143	2.56	.217	2.120	1.111	-3.23		1.478	1.270	-3.33	0.045	-0.150	1.498	Shock diagram			-.083	.828	1.678	
S_{II}					1.398	.911	1.139	2.48	.221	2.437	.987		-1.507	1.388	1.218	Interpolation between M_{II} and N_{II}			1.605	Interpolation between M_{II} and N_{II}			-.209			
11		S_{II}	1.648	-2.18	1.410	.933	1.141	2.50	.221	2.470	.956		-1.505	1.388	1.225	From boundary condition			1.646	-1.506	0.007	-0.011	-.220			
N_{II}	11		1.495	-.083	1.415	.790	1.142	2.55	.221	2.120	1.111	-3.23		1.478	1.270	-3.33	0.045	-0.150	1.496	Shock diagram			-.083			

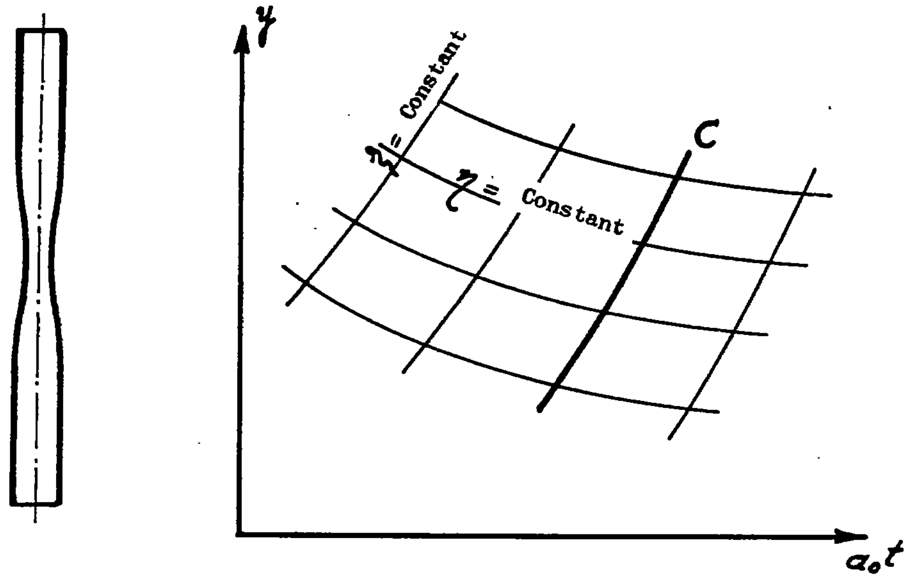


Figure 1.- Curvilinear coordinate system ξ , η .

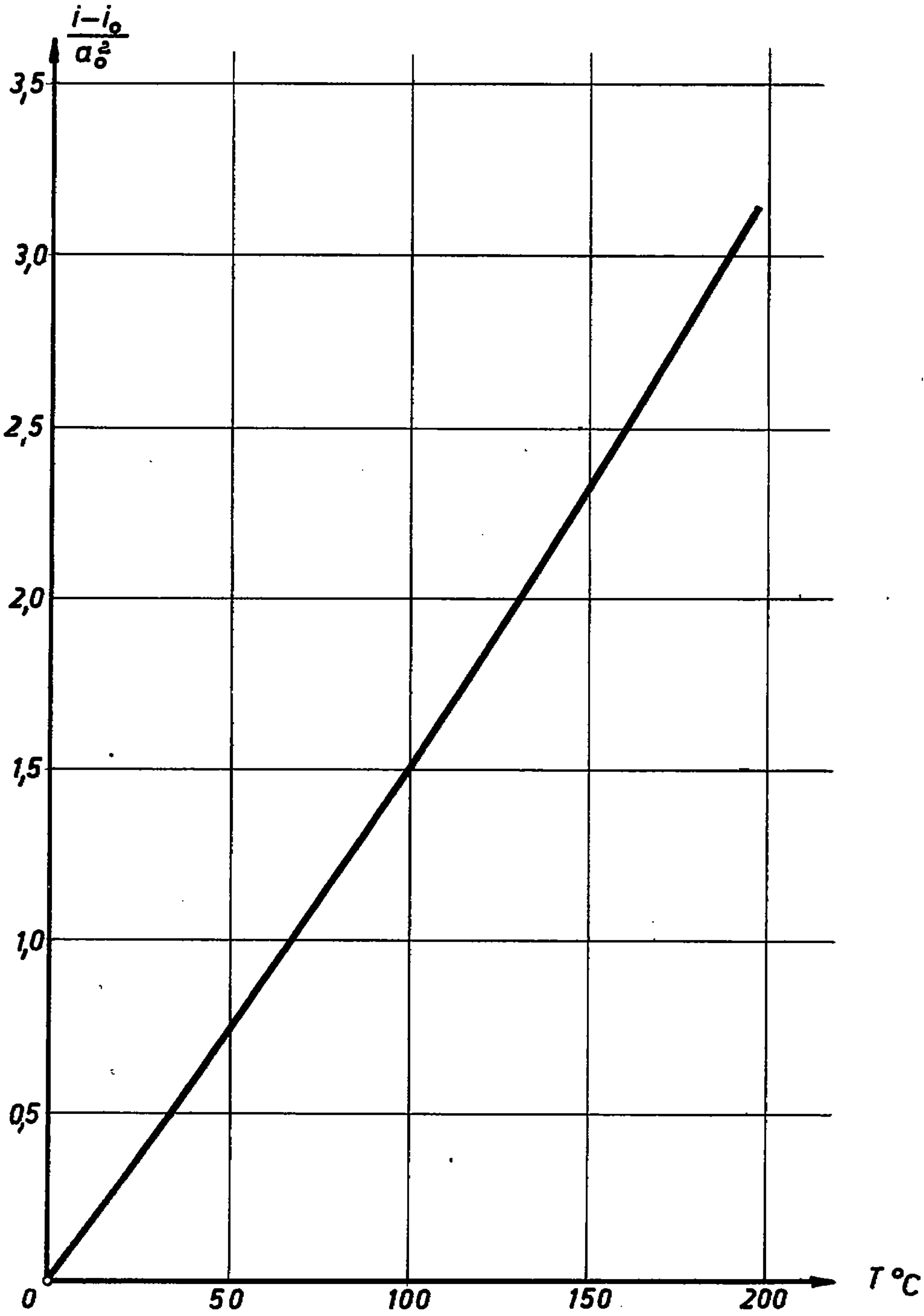


Figure 2a.- Relation between i and T for CO_2.

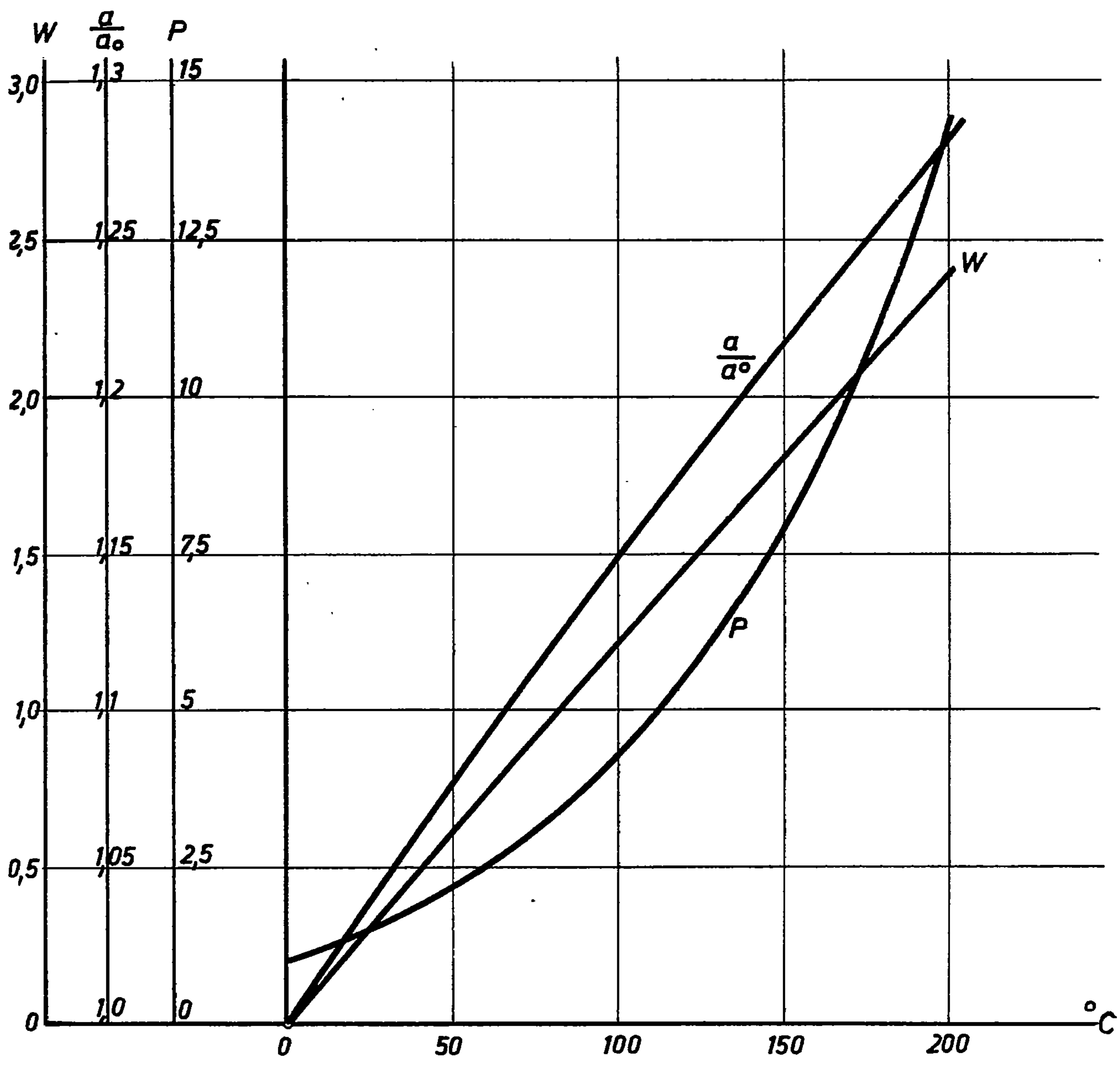


Figure 2b.- $\frac{a}{a^o}$; P ; W as functions of the temperature for CO_2.

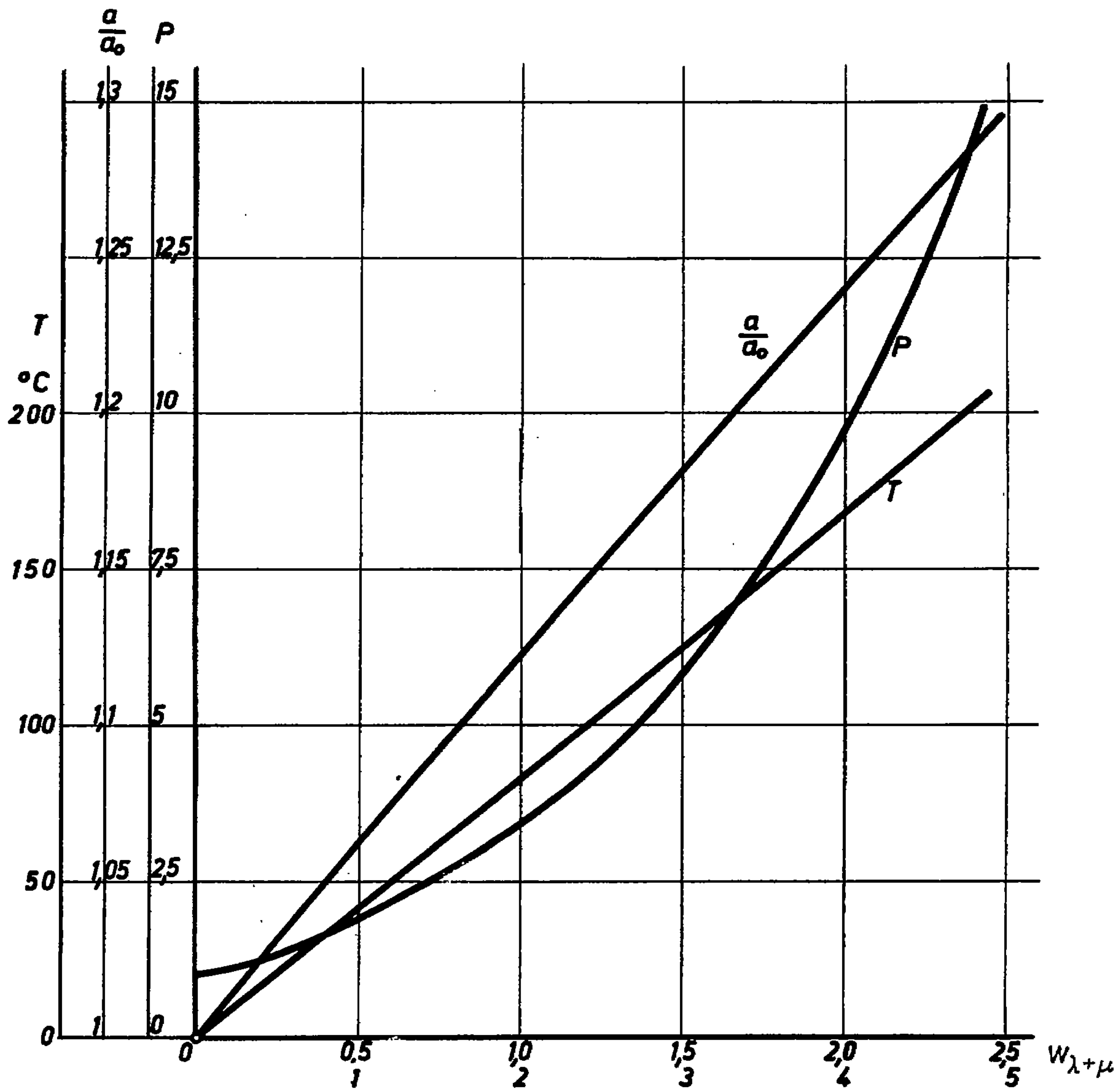


Figure 3.- $\frac{a}{a_0}$; P; T as functions of W, or $\lambda + \mu$, for CO_2.

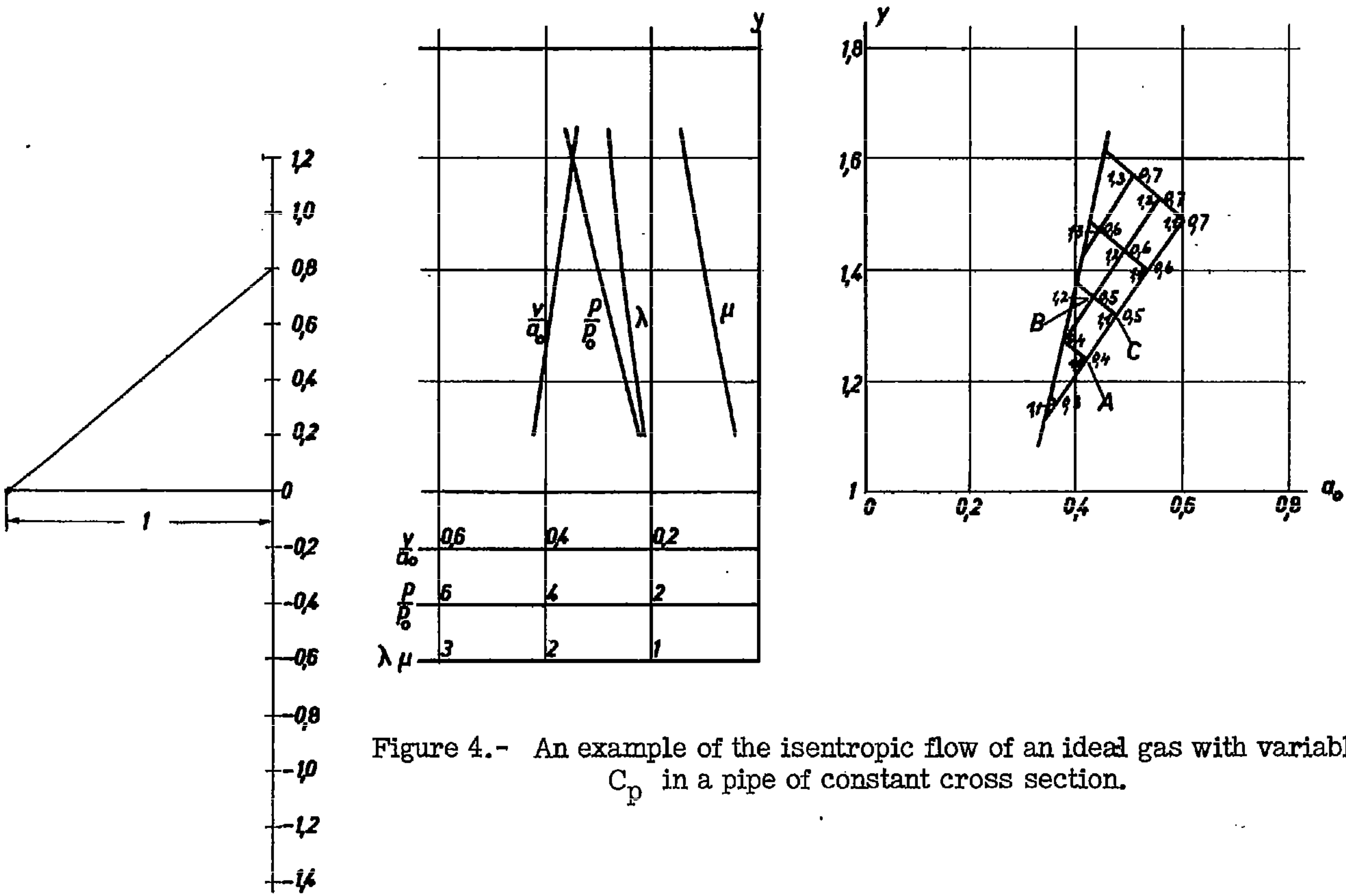
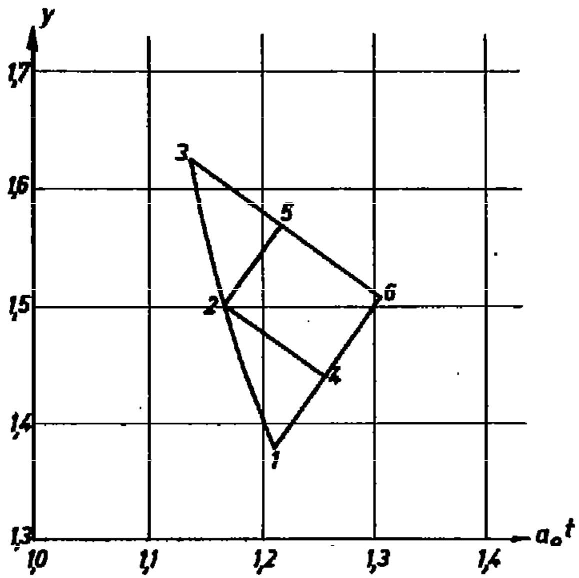


Figure 4.- An example of the isentropic flow of an ideal gas with variable C_p in a pipe of constant cross section.



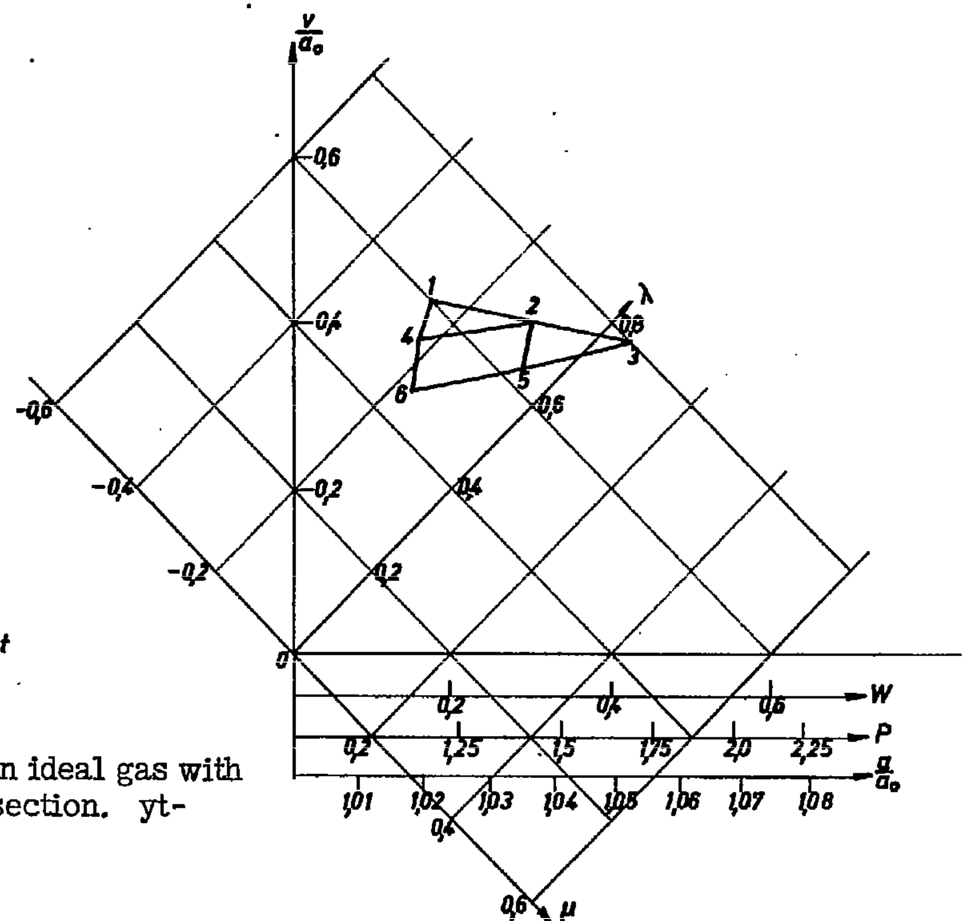


Figure 5.- An example of the isentropic flow of an ideal gas with variable specific heat and varying pipe cross section. yt- and λ μ-chart.

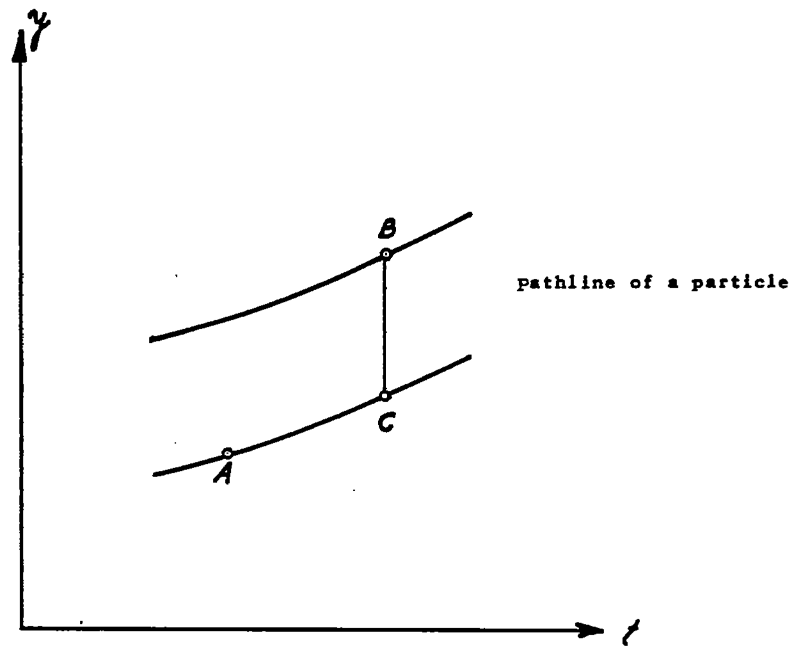


Figure 6.- The physical significance of ψ.

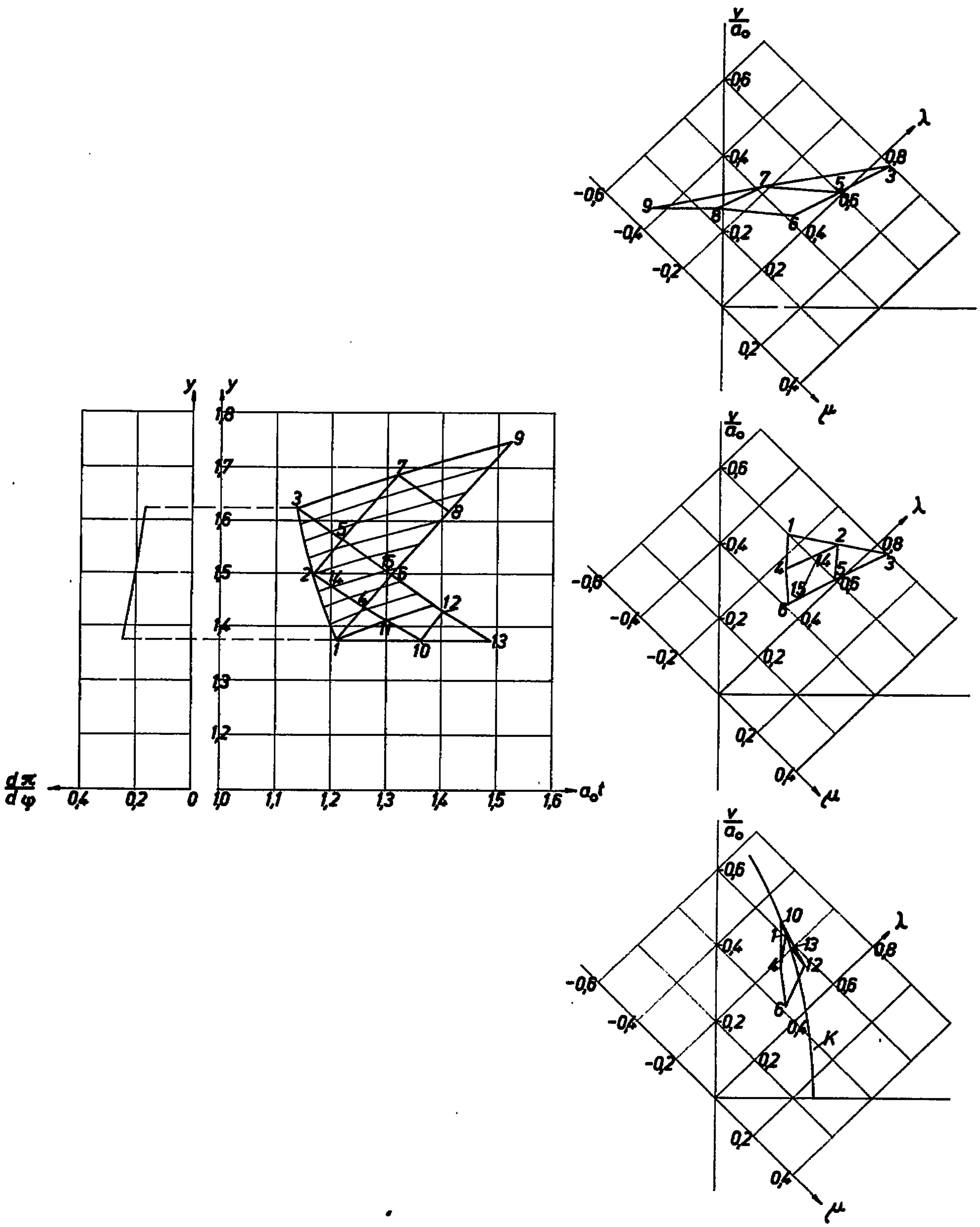


Figure 7.- Any flow of an ideal gas with variable specific heat. Treatment of boundary conditions.

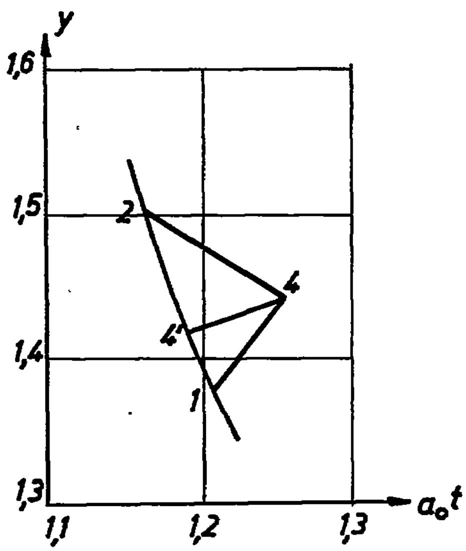


Figure 8.

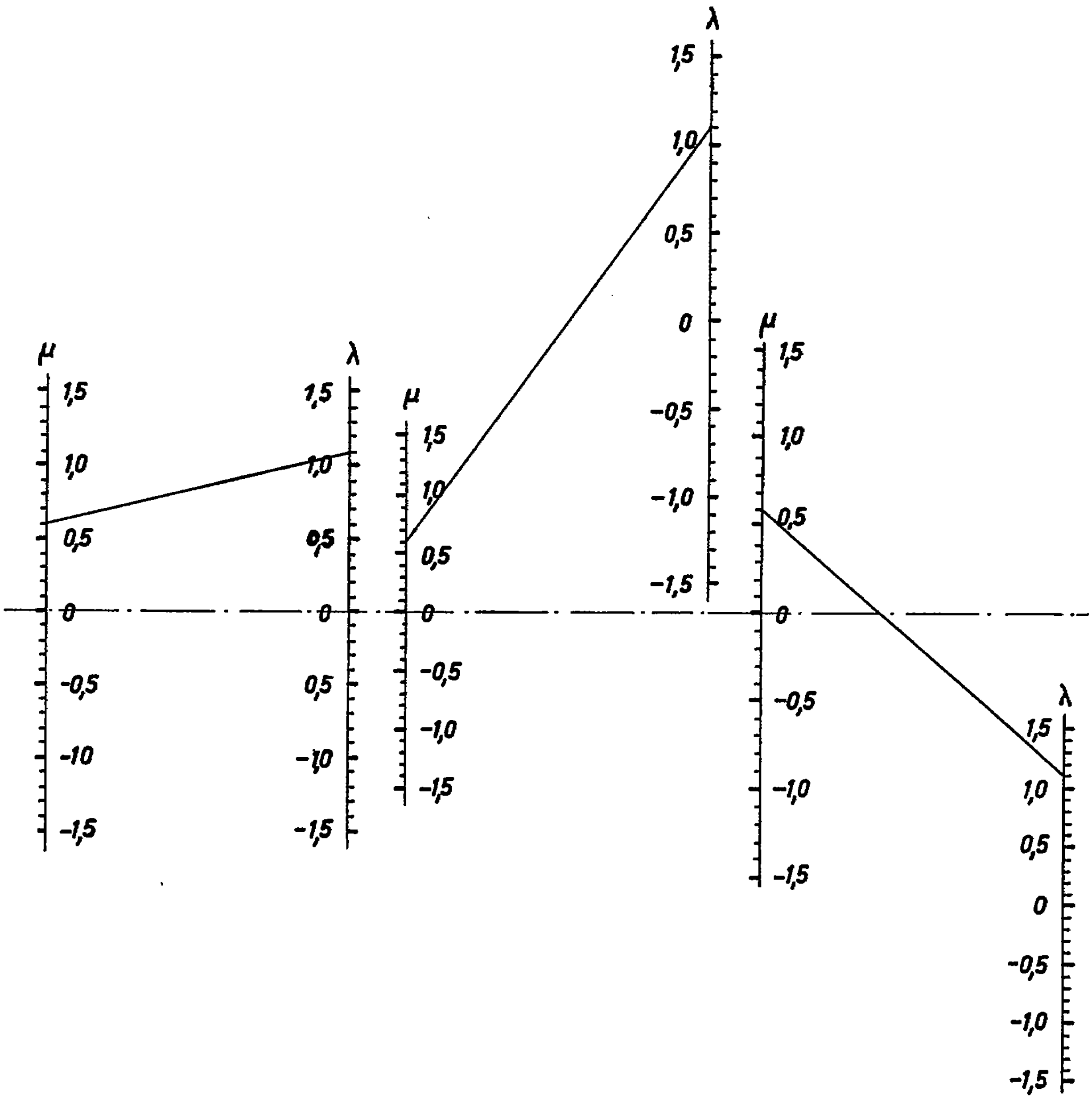


Figure 9.

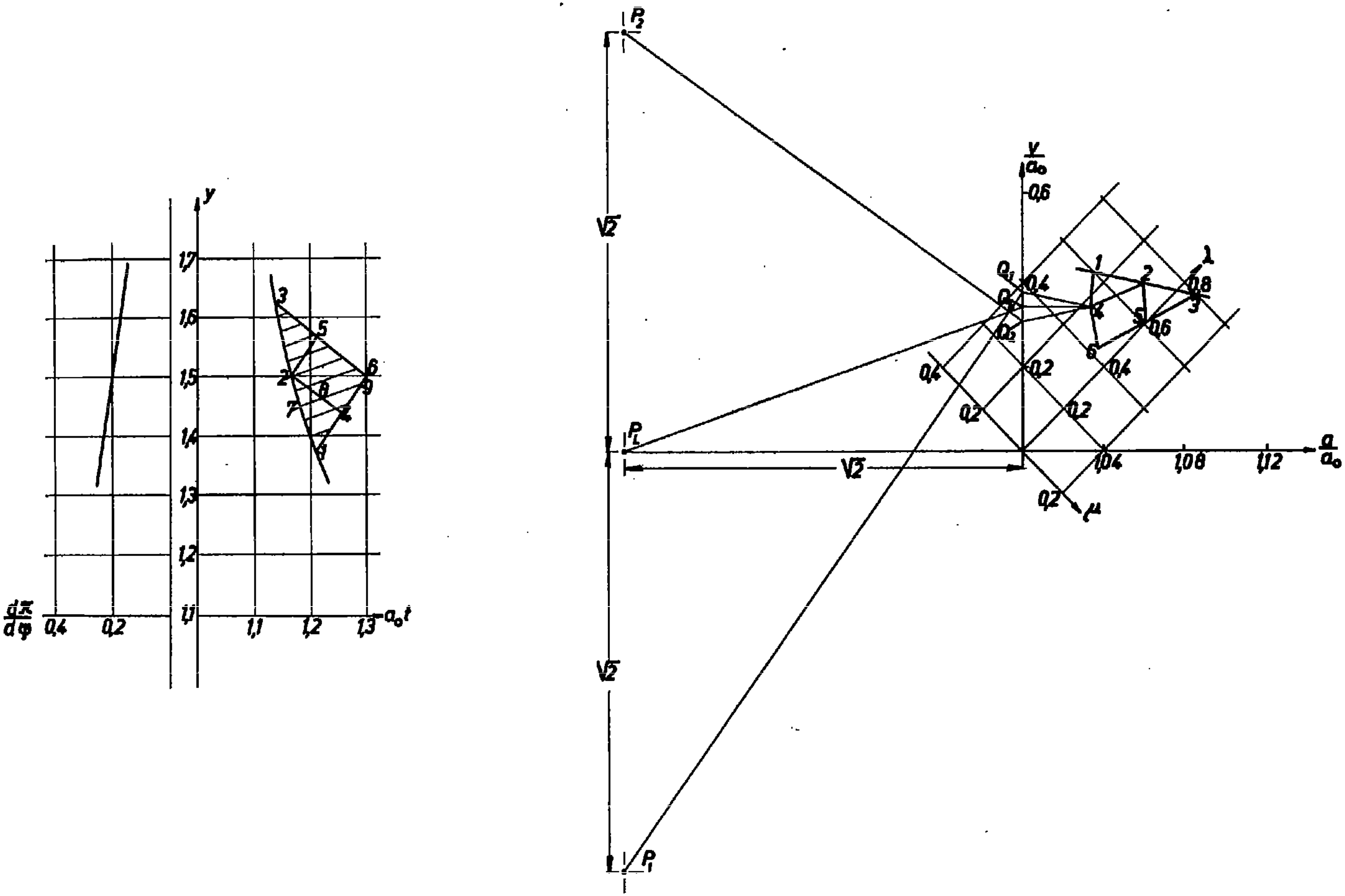


Figure 10.- Flow of an ideal gas with constant K = 1.405. Determination of the direction of the Mach waves and life lines from the λ μ-chart.

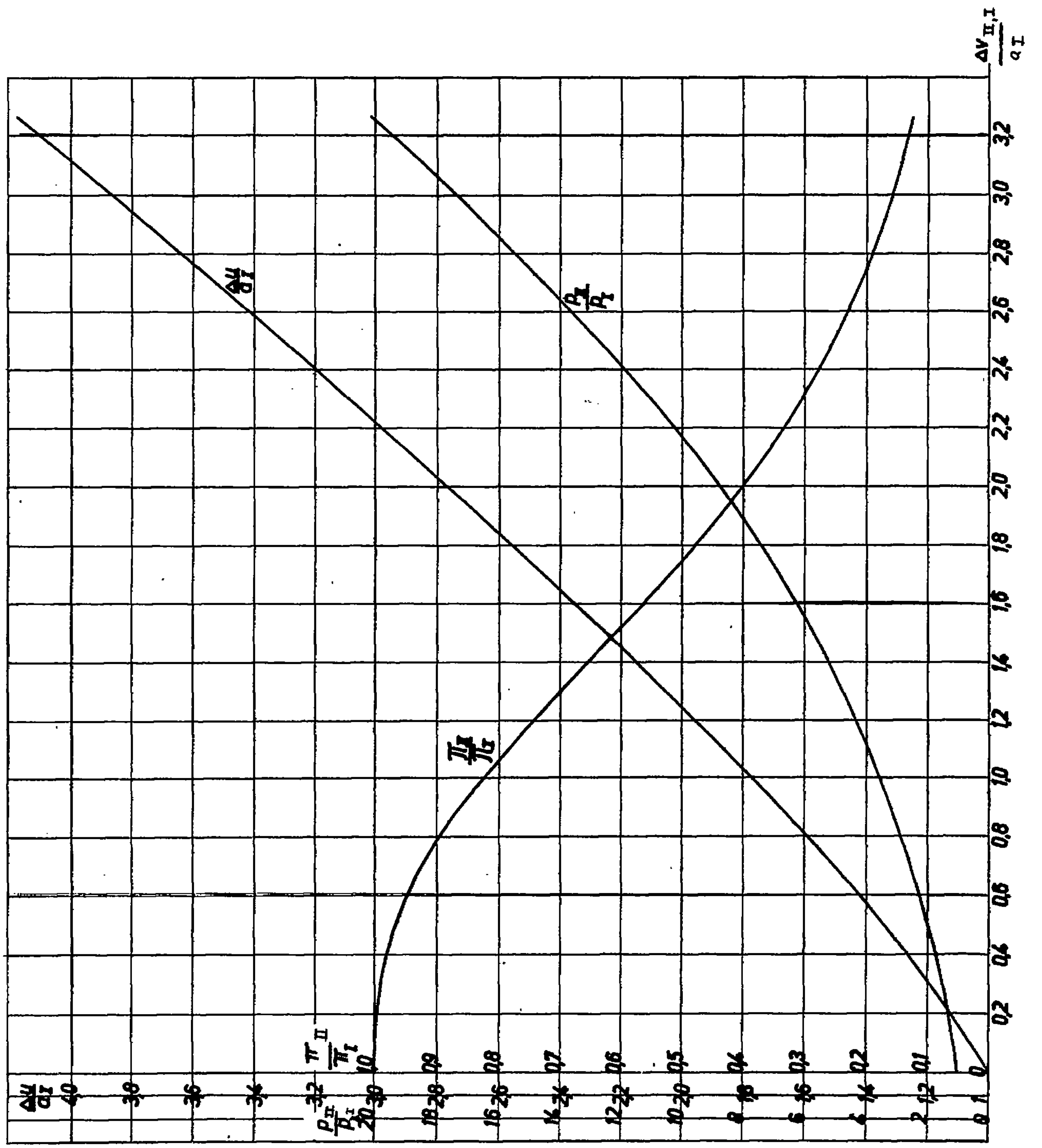


Figure 11.- Characteristics of compression shocks K = 1.405.

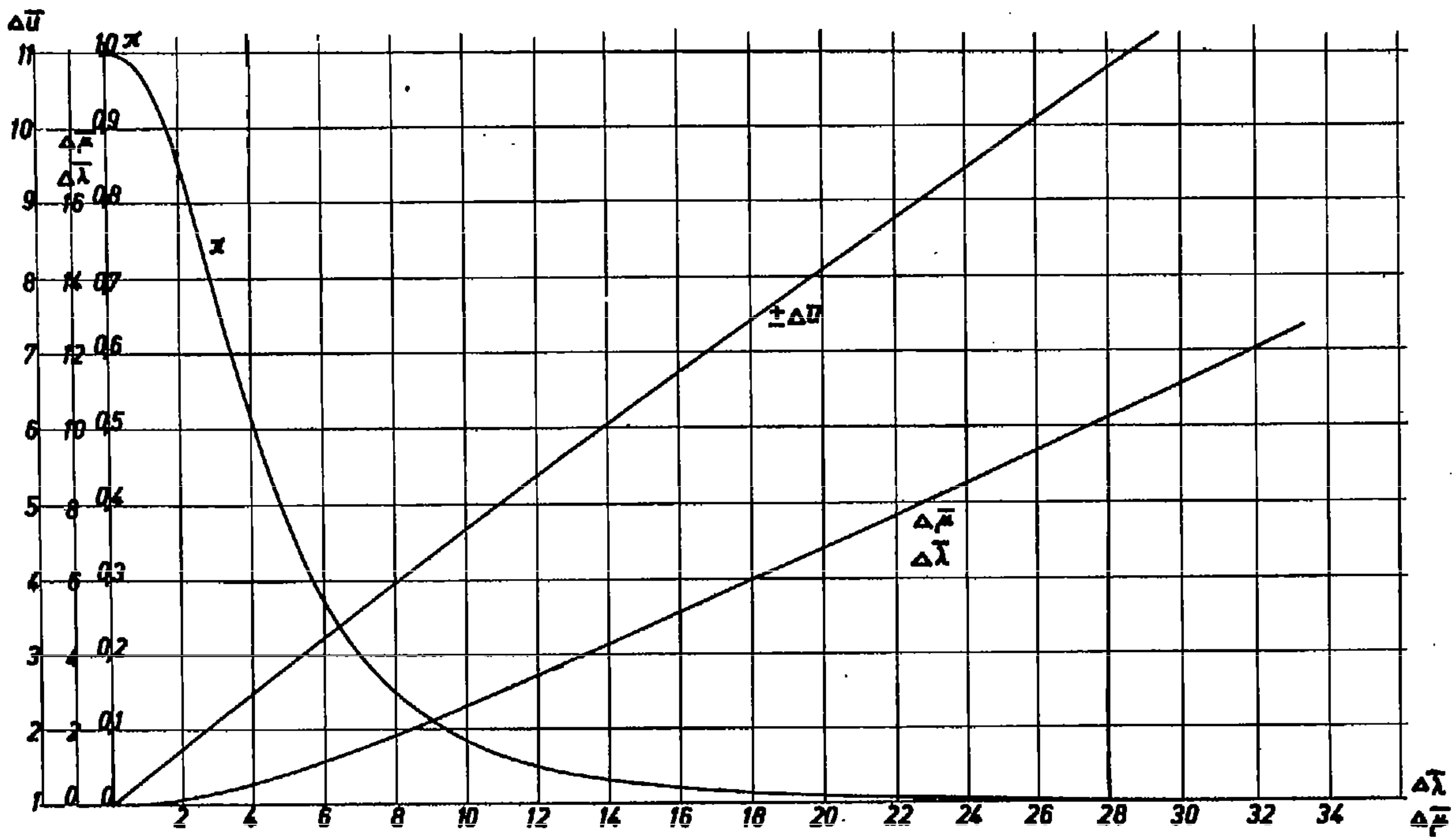


Figure 12a.- Characteristics of compression shocks K = 1.405.

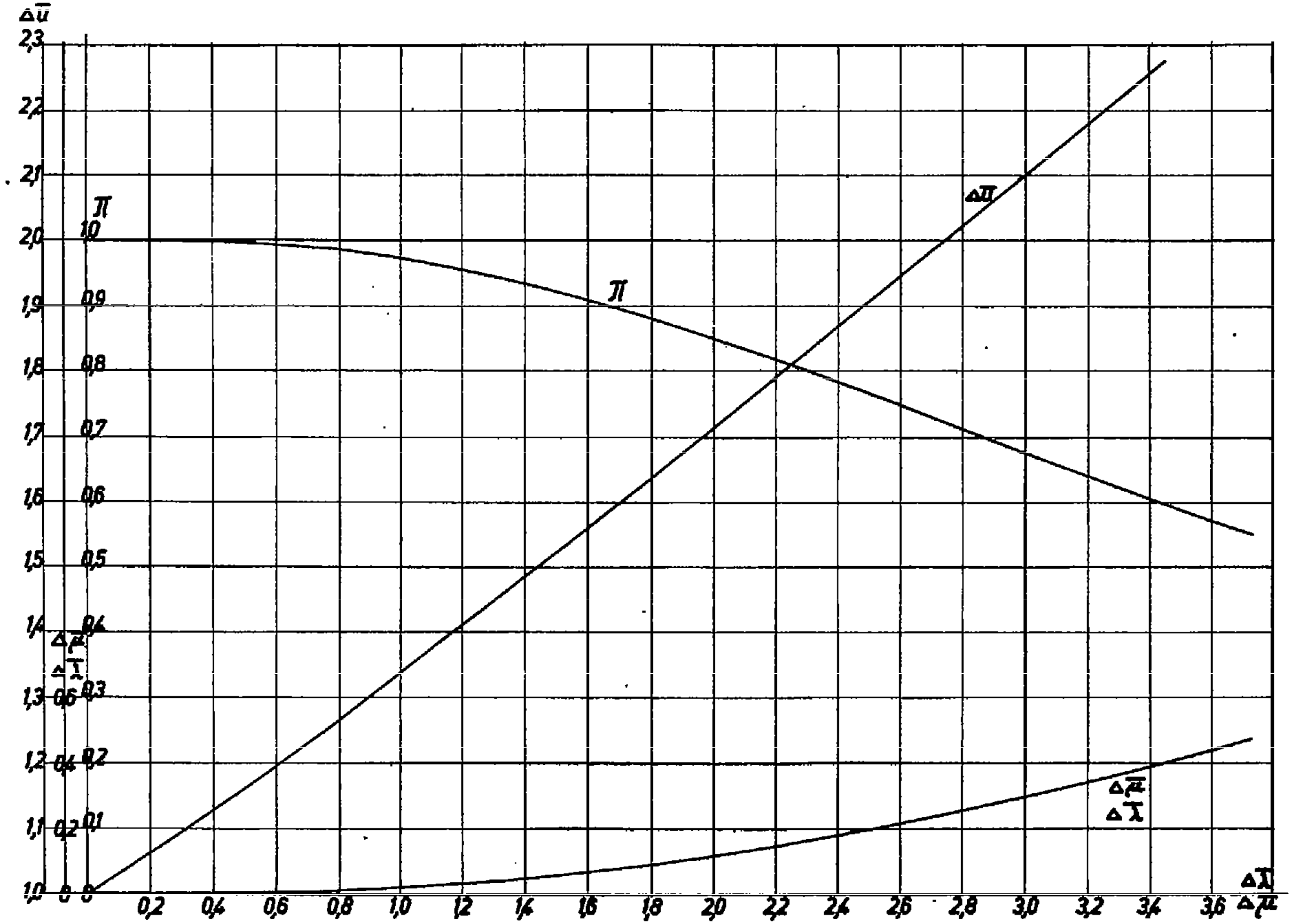


Figure 12b.- Characteristics of compression shocks K = 1.405.

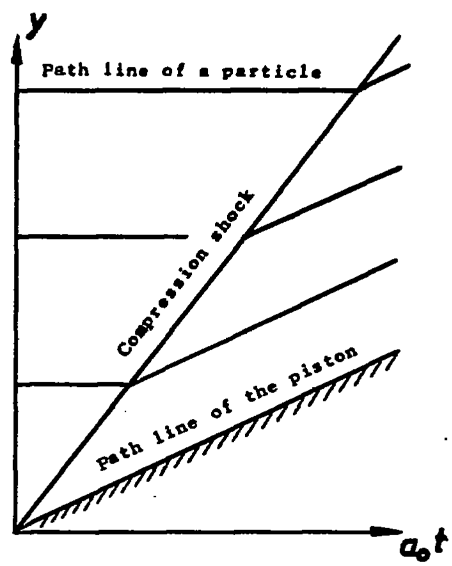

y
Path line of a particle
Compression shock
Path line of the piston
$a_o t$

Figure 13.

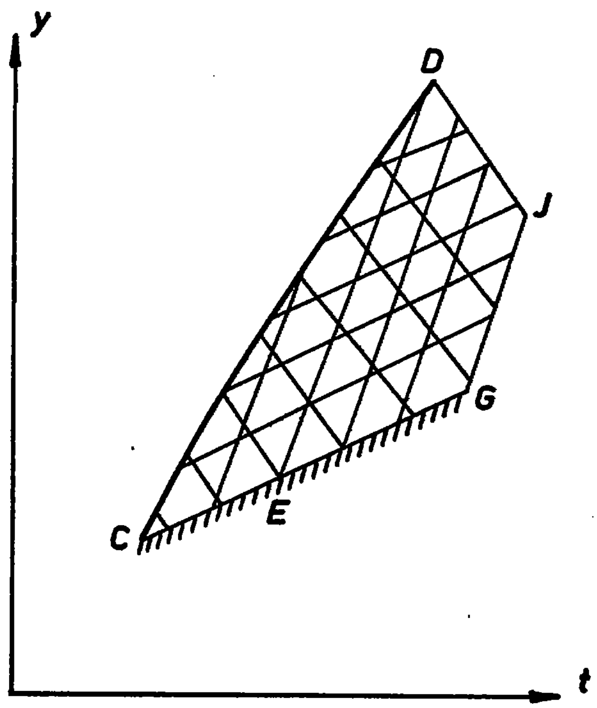


Figure 14a.

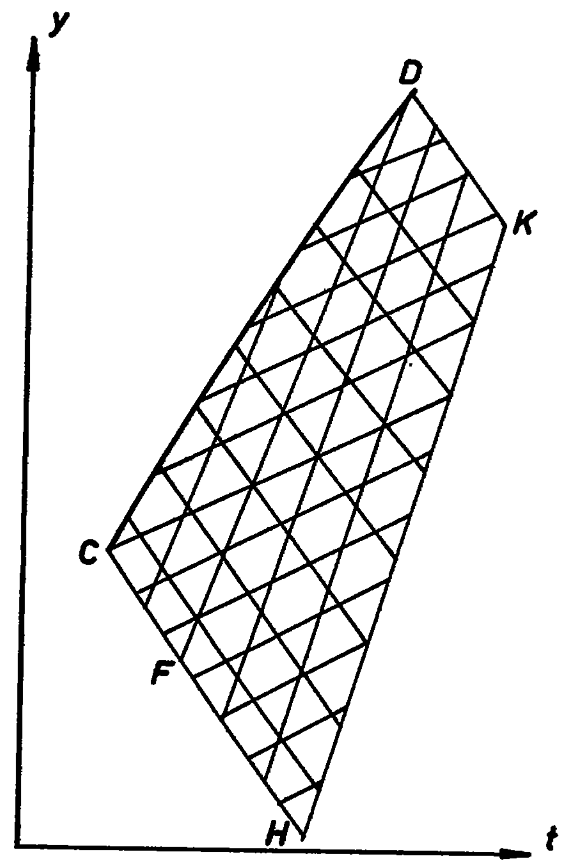


Figure 14b.

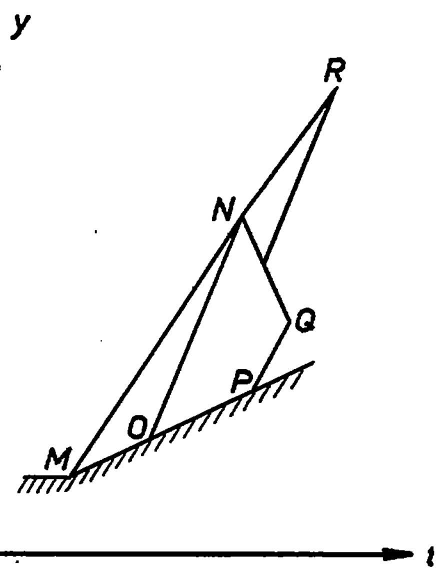


Figure 15.

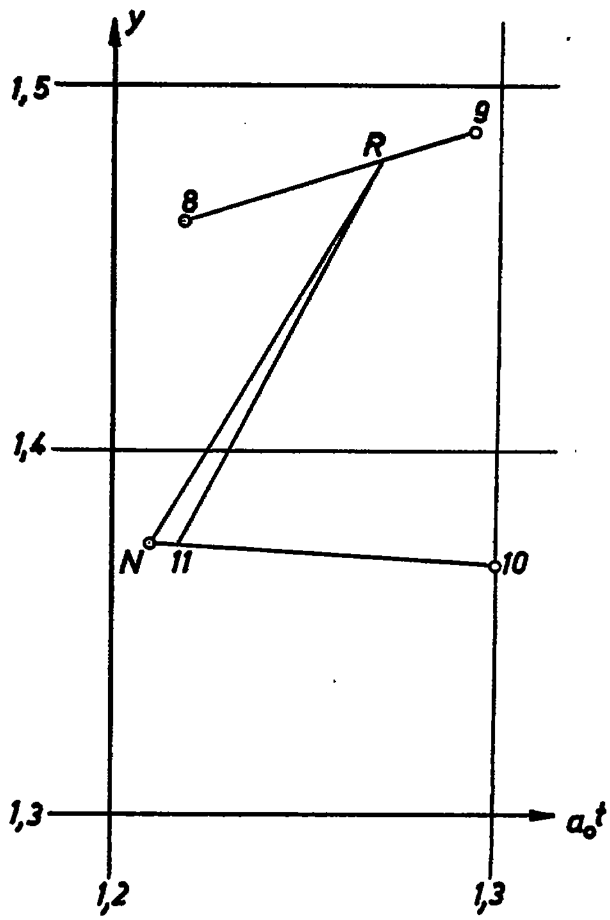


Figure 16a.

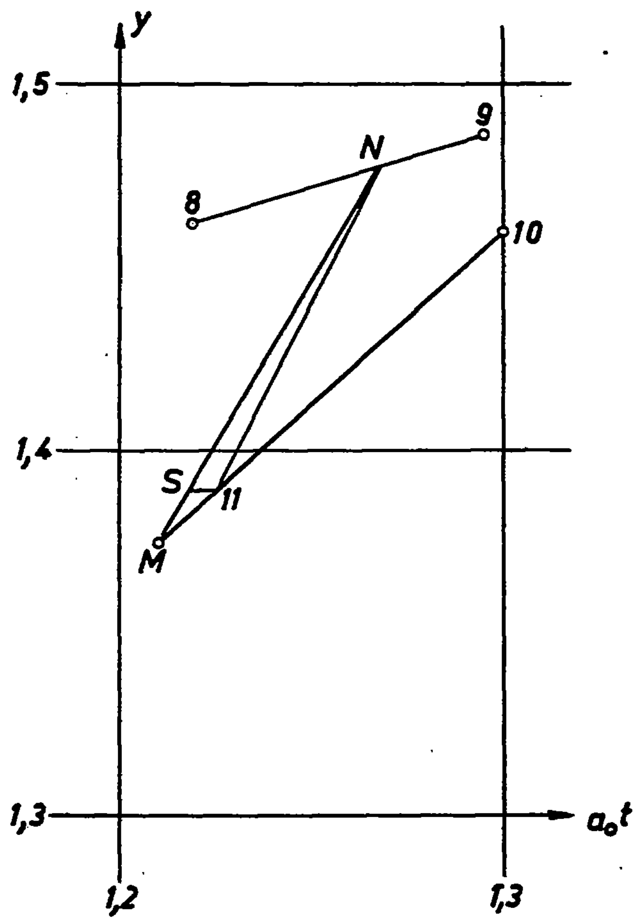


Figure 16b.